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Complex Numbers

Complex numbers are the most general numbers used in algebra. Any number that can be expressed in the form $a + bi$, where a and b are real numbers and $i^2 = -1$ is a complex number. This may be confusing to anyone unfamiliar with this definition, since, for example, calculators cannot compute the square root of -1 . This is because calculators do only real arithmetic. The existence of i is of fundamental importance to mathematics. Engineers often use the “ j ” notation, $j = i$.

Complex numbers are not an abstraction of theoretical mathematics. Without complex numbers, many polynomial equations would have no solution. One very simple example is $x^2 = -1$. The solution to this equation ($x = i$) cannot be represented by a real number. Complex numbers have very many applications in applied math, physics, and engineering.

A complex number can be thought of as a two-dimensional vector (a, b) , where a is the **real part** and b is the **imaginary part**. The term “imaginary” is an unfortunate misnomer left over from the seventeenth century, when mathematicians were still uncomfortable with the concept of complex numbers. The imaginary part is every bit as real as the real part of the complex number. Sometimes complex numbers are represented in the **standard form** $a + bi$. Another common representation is the **polar or trigonometric form**

$$(a, b) = z = r[\cos(\theta) + i \sin(\theta)]$$

The **magnitude** of a complex number is the square root of the sum of the squares of its real and imaginary part:

$$|(a, b)| = (a^2 + b^2)^{0.5}$$

The **conjugate** of the complex number (a, b) is $(a, -b)$. The complex conjugate is sometimes denoted as $(a, b)^*$, where

$$(a, b)^* = (a, -b)$$

Complex Arithmetic. The arithmetic of complex numbers is defined as follows:

addition	$(a, b) + (c, d) = (a + c, b + d)$
subtraction	$(a, b) - (c, d) = (a - c, b - d)$
multiplication	$(a, b) * (c, d) = (ac - bd, ad + bc)$
division	$(a, b)/(c, d) = (ac + bd, ad + bc)/(c^2 + d^2)$

Powers of i . Some important identities involve the powers of i :

$$\begin{aligned} i^1 &= i^5 = i^9 = \dots = i \\ i^2 &= i^6 = i^{10} = \dots = -1 \\ i^3 &= i^7 = i^{11} = \dots = -i \\ i^4 &= i^8 = i^{12} = \dots = +1 \end{aligned}$$

Note that when the powers of i are simplified, they cycle in steps of four.

Powers and Roots of Complex Numbers. Both the n th power and the n th root of a complex number are also complex numbers, which are best represented in **polar form**:

nth power	$\{r[\cos(\theta) + i \sin(\theta)]\}^n = r^n[\cos(n\theta) + i \sin(n\theta)]$
nth root	$z^{1/n} = r^{1/n}[\cos(\theta/n) + i \sin(\theta/n)]$

For both the n th power and the n th root, the angle θ must be evaluated modulo 360° . This is a direct consequence of the periodicity of the sine and cosine functions. That is, suppose θ is the angle (in degrees) that resolves an n th root (or power) of z . Then for any integer, k , the larger angle $\theta + k*360$ also resolves the root (or power).

This ambiguity leads to the definition of a **principal n th root** of a complex number. A **principal n th root** of z is a root with a polar angle between 0 and 360 degrees:

$$0 \leq \theta/n \leq 360$$

There are always n unique principal n th roots of a complex number. For example, there are three principal cube roots of $2i$:

$$\begin{aligned} &2^{1/3}[\cos(30) + i \sin(30)] \\ &2^{1/3}[\cos(150) + i \sin(150)] \\ &2^{1/3}[\cos(270) + i \sin(270)] \end{aligned}$$

All other cube roots are redundant.

See Functions: Cadd(), Cdiv(), Cmag(), Complx(), Conjg(), Cmul(), CscalR(), Csqrt(), Csub().