1

Integral Solutions to Linear Programs

The problems we have studied so far have one property in common: there exist optimal solutions that are integral if the data (in some cases some part of the data) are integral. (If one replaces the word "integral" by the word "rational" then this statement is true for general LP). We now look at the conditions that allow this to happen. In this section we assume that A is an integral matrix and that no constraint is redundant.

Let $S = [x : Ax = b; x \geq 0]$ and $S^* = [x : Ax \leq b; x \geq 0]$.

Theorem 1 *The following statements are equivalent:*

1. For every basis B of A $\det B = \pm 1$
2. All extreme points of S are integral for all integral b
3. B^{-1} is integral for all bases B of A.

Proof: We prove that (1) $\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(1)

(1)$\Longrightarrow$(2): Consider a feasible basis B. The corresponding basic feasible solution x^B satisfies the relation $Bx^B = b$ and is the unique solution to it. By Cramer's rule $x^B_j = \frac{\det\ D_j}{\det\ B}$ where D_j is the matrix obtained when we replace the j^{th} column of B by b. Since D_j is an integral matrix and $\det B = \pm 1$ we have the desired result.

(2)$\Longrightarrow$(3): Let $b^i = By^i + e_i$ where y^i is integral and satisfies $y^i + B^{-1}_{.i} \geq 0$ and e_i is the i^{th} unit vector. Consider the system S with this b^i. Clearly the basis B is feasible and the corresponding extreme point is given by $x^B = y^i + B^{-1}_{.i}$ and by hypothesis is integral and hence $B^{-1}_{.i}$ is integral.

(3)$\Longrightarrow$(1): Since B and B^{-1} are integral the result follows from the fact that $\det I = 1$.

Let A be a matrix of rank k. We say that it is *unimodular* if every submatrix B of size k has a determinant equal to ± 1 or 0.

Theorem 2 *The following statements are equivalent:*

1. Every nonsingular submatrix of A has a determinant whose value is 0, or ± 1
2. Extreme points (basic feasible solutions) of S^* are integral for all integral b
3. Every nonsingular submatrix of A has an integral inverse.

Proof: Let $A' = [A, I]$ and use the equivalence of conditions of theorem 1 and the corresponding ones of this theorem.

A matrix all of whose submatrices have determinant equal to 0 or ± 1 is called a *totally unimodular* matrix.

Corollary 3 *A is totally unimodular (t.u) iff each of (i)A^t; (ii) B; (iii)C; (iv)$[A,\ I]$; (v)$\left[\begin{smallmatrix} A \\ I \end{smallmatrix}\right]$; (vi)$A^{-1}$ if it exists is totally unimodular. Here B and C are defined as follows: $B_{.j} = \pm A_{.j}$ and C is a matrix whose distinct columns (rows) from a subset of columns (rows) of A.*

Proof: All statements except (vi) are straight forward. To show (vi) consider the set;

$\hat{S} = [x, s : A^{-1}x + Is = b; x \geq 0; s \geq 0]$

$= [x, s : As + Ix = Ab = b'; x \geq 0; s \geq 0]$

Since A is $t.u.$, extreme points of $\hat{S}$ are integral for all integral b' and hence all b (since A is nonsingular) and hence A^{-1} is $t.u.$

Theorem 4 *Let A be matrix with $a_{i,j} = 0/\pm 1$ and $\sum_i |a_{i,j}| \leq 2\ \forall\ j$. This condition implies that the number of nonzeroes in each column is at most two. Then A is t.u. iff the rows can be partitioned into two sets R_1 and R_2 $\ni$:(i)if a column has two nonzeroes of the same sign then there is one in each set.(ii)if a column has two nonzeroes of opposite sign then both are in the same set.*

Proof: It is easy to verify that submatrices of such matrices satisfy the same condition. *Such a property is said to inherited.* Hence it suffices to show this for square matrices. The proof is by induction on the size of the matrix. It is trivially true for $k = 1$. Assume that it is true for $k - 1$. Let A be a $k \times k$ matrix. If there is a column with exactly one nonzero then by cofactor expansion and the induction hypothesis the theorem follows. If every column has two nonzeroes then the matrix is singular since sum of the rows in R_1 equals the sum in the set R_2. For the only if part it is easy to verify that if the condition does not hold then there is a submatrix whose determinant equals two. Hence the theorem.

Corollary 5 *E is t.u.*

Corollary 6 *The constraint matrix of the transportation problem is t.u.*

Theorem 7 *For an integral matrix with at most two nonzeroes in each column which are ± 1 the condition of the previous theorem is necessary.*

Definition 1 *A matrix A is said to* Dantzig property *if $Ax = A_{.j} \Longrightarrow x$ is a vector all of whose components are 0, or ± 1.* ***G.B. Dantzig*** *showed that this property was valid for the constraint matrix of the transportation problem.*

Theorem 8 *A t.u $\Longrightarrow A$ has Dantzig property.*

Proof: Suppose not. Let $Ax = A_{.j}$; and x is not such a vector. We may assume that A is nonsingular without loss. Consider the matrix $A^{-1}[A, A_{.j}] = [I, x]$. Since there is a component of x which is not 0 or ± 1, there is a submatrix of this matrix whose determinant is not 0 or ± 1. Taking the corresponding submatrix of $[A, A_{.j}]$ provides the desired result.□

Since the Dantzig property does not require the elements to be 0 or ± 1, we need additional assumptions for the converse.

Theorem 9 *Let A be a $m \times n$ matrix whose rank is m with the Dantzig property. Let B be a basis; then $B^{-1}A$ is t.u.*

Proof: Dantzig property for A is the same as the property for $B^{-1}A$ which is of the form $[I, \bar{A}]$. Dantzig property for $[I, \bar{A}]$ is equivalent to $t.u$ of this matrix.□

Theorem 10 *Let A be a $m \times n$ matrix whose rank is m. Unimodularity of A is equivalent to t.u. of $B^{-1}A$ and $|detB| = \pm 1$ for all basis B of A.*

Proof: Follows from the unimodularity of $B^{-1}A$ and the form of $B^{-1}A = [I, \bar{A}]$.□

The most important result in this area , in our opinion, is a result attributed to **R.E.Gomory** and stated and proved in a paper by **P.Camion**. We give a slightly different proof. It states that the determinant of a minimal $non - t.u.$ matrix whose entries are 0, or ± 1 is ± 2.

Lemma 11 *Let G be square, $0/\pm 1$ matrix which is not t.u. but all of whose submatrices are t.u. Then: (i)$|\det G| = 2$; (ii) All entries in G^{-1} are $\pm 1/2$; (iii) $e^t G \equiv Ge \equiv 0 \pmod 2$; (iv) $e^t Ge \equiv 2 \pmod 4$. Matrices satisfying (iii) are said to be* Eulerian.

Proof: Let $d = |\det G| \geq 2$. Since all submatrices are $t.u.$,

$H = G^{-1} = \frac{1}{\det G} G^{\#}$ where $G^{\#}$ is the cofactor matrix for G and hence is a $0/\pm 1$ matrix. Hence all elements of H are 0 or $\pm \frac{1}{d}$. First we will show that there are no zeroes.

Suppose $H_{j,k} = 0$. Let $\hat{H}$ and $\hat{G}$ be matrices obtained from H and G respectively by deleting column j. Then $\hat{G}\hat{H}_{.k} = GH_{.k} = e_k$. $\hat{G}$ is $t.u.$ by assumption and has independent columns and e_k is integral and hence $\hat{H}_{.k}$ (and hence $H_{.k}$) is integral. However, each element of $H_{.k}$ is 0 or $\pm\frac{1}{d}$ with $d \geq 2$ and the vector $H_{.k} \neq 0$. This is impossible. Hence, the assumption that $H_{j,k} = 0$ is invalid. Thus, each element of H is $\pm\frac{1}{d}$. Now we show that $d = 2$.

Let $G = \begin{bmatrix} K & \cdot \\ y & \cdot \end{bmatrix}$ where K is a $(m-1) \times (m-1)$ matrix and G is an $m \times m$ matrix. Since G^{-1} has no zeroes, and K is a cofactor of G, K is $t.u.$ and nonsingular. Hence x in $Kx = e_1$ is a vector all of whose components are 0, or ± 1. Let $G \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 \\ \nu \end{bmatrix}$; $v = yx$ and hence is an integer. $\begin{bmatrix} e_1 \\ \nu \end{bmatrix} = G(G^{-1}_{.1} + vG^{-1}_{.m})$; Since G is nonsingular $\begin{bmatrix} x \\ 0 \end{bmatrix} = G^{-1}_{.1} + vG^{-1}_{.m}$. Recall that all entries in G^{-1} are $\pm 1/d$ and two columns of G^{-1} are linearly independent we must have the following "picture" of a 2×2 submatrix from the columns of $G^{-1}_{.1}$ and $G^{-1}_{.m}$: $\frac{1}{d}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$. Then, $(1/d) + (v/d) \equiv 0$ $(mod 1)$ and $(1/d) - (v/d) \equiv 0$ $(mod 1)$ since x has only components whose value is 0 or ± 1. Adding these we get: $2/d \equiv 0$ $(mod 1)$. Since $d \geq 2$ we have the result that $d = 2$. We have completed the proofs of (i) and (ii).

Since all elements of H are $\pm 1/2$, all elements of $2H$ are ± 1. Let $y = 2H_{.1}$. Then it is clear that $Ge \equiv Gy$ $(mod\ 2)$. $Gy = 2e_1 \equiv 0$ $(mod\ 2)$ and hence the result follows. Row *eulerianness* follows in a similar manner. This proves (iii).

From (iii) $e^tGe \equiv 0$ or $2(mod\ 4)$. We will exclude the first possibility by a lemmas below.

Lemma 12 *Let B be square, column eulerian. Then* det $B \equiv 0$ $(mod\ 2)$.

Proof: If B is nonsingular, let $Be = 2y$, y integral. Then $e = 2B^{-1}y = 2(\text{adj}\, B/det\ B)y$. Hence $(\det B)e = 2\,(\text{adj}\ B)y$ which is even. Hence det $B \equiv 0$ $(mod\ 2)$.

Lemma 13 *Let B be square, eulerian with $e^tBe \equiv 0$ $(mod\ 4)$. Then* det $B \equiv 0$ $(mod 4)$.

Proof: Without loss assume det $B \neq 0$. Let $\hat{B}$ be defined by $\hat{B}_{1.} = \sum_i B_{i.}$; $\hat{B}_{i.} = B_{i.}$ for $i \neq 1$. Every entry in $\hat{B}_{1.}$ is even since B is row eulerian. Since B is column eulerian, so is $\hat{B}$ and hence $\hat{B}e = 2y$ with y integral. $(\det \hat{B})e = 2(\text{adj}\ \hat{B})y$; adj $\hat{B}$ has even entries in all rows except possibly the first since the first row of $\hat{B}$ has only even entries. Also, the first component of y is even since the first element of $\hat{B}e = e^tBe \equiv 0$ $(mod\ 4)$. Hence (adj $\hat{B})y \equiv 0$ $(mod\ 2) = 2z$ with z integral. Hence $(\det \hat{B}).e = 2 \times 2z \equiv 0$ $(mod\ 4)$. Hence the lemma.