

# LINEAR PROGRAMMING AND EXTENSIONS

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# Chapter 1

## Geometry of LP

The most important concepts in this area are *convexity of sets and functions*; *topological properties of closed and open sets*; the notion of *bounded and unbounded sets*, etc. We will define the terms used and show the relationships first in pure geometric terms and then relate these to the corresponding algebraic notions.

Convex Subsets in  $\mathbf{R}^n$  : A set  $X \subset \mathbf{R}^n$  is said to be *nonconvex* if there are two points in the set such that the line segment joining them is not entirely in the set; equivalently, if

$$\exists x \in X, y \in X; \lambda \in (0, 1) \ni \lambda x + (1 - \lambda)y \notin X$$

A set is *convex* if it is not nonconvex. The set  $L = \{z : z = \lambda x + (1 - \lambda)y\}$  is the *straight line* passing through points  $x$  and  $y$ . If in addition  $\lambda \in [0, 1]$ , then  $L$  is called the *line segment* with  $x$  and  $y$  as its terminals. “Clearly”,  $L$  is a convex set (under either condition). When  $\lambda \in [0, 1]$ , a point  $z \in L$  is called a *convex combination* of  $x$  and  $y$  — and is a point somewhere between  $x$  and  $y$  on the line joining the two. If  $S = \{x^1, x^2, \dots, x^n\}$  is a set of points, then  $z = \sum_j \lambda_j x^j$  is a convex combination of the points in  $S$  if  $\lambda_j \geq 0$ ; and  $\sum_j \lambda_j = 1$ . The set of all such points  $z$  is called the *convex hull* of the set  $S$ . (*Indeed this notion can be suitably extended even when the set  $S$  is not finite; of course, the sum would be replaced by an integral*).

### Facts:

1. If  $S$  and  $T$  are convex sets, then so is  $S \cap T$ . Indeed, if  $S_\alpha$  is a collection of convex sets for  $\alpha \in A$ , then  $\bigcap_{\alpha \in A} S_\alpha$  is a convex set. *In order to avoid special cases, it is customary to assert (by default) that  $\phi$  is a convex set.*
2. If  $S$  is convex,  $x^i \in S$ ;  $1 \leq i \leq n$  and  $\lambda_i \geq 0$ ;  $\sum_i \lambda_i = 1$ , then  $\sum_i \lambda_i x_i \in S$ . (The proof of this is by induction on  $n$ . What would you do if the number of points selected to form the convex combination is not finite?)

3.  $H = \{x : cx = \alpha\}$  is called a *hyperplane*.  $H^- = \{x : cx \leq \alpha\}$ ,  $H^{--} = \{x : cx < \alpha\}$ ,  $H^+ = \{x : cx \geq \alpha\}$  and  $H^{++} = \{x : cx > \alpha\}$  are called *halfspaces*.  $H^-$  and  $H^+$  are *closed* and the other two are *open*. All of these are convex sets. Intersection of any number of these (*an example would be the set of feasible solutions of a linear program*) is a convex set; if the number of these is finite the set is said to be *polyhedral*. Intersection of a finite number of hyperplanes is called an *affine set*. Affine sets are of the form  $\{x : Ax = b\}$ ; the dimension of the subspace  $\{x : Ax = 0\}$  is the dimension of the above affine set.
4.  $D = \{x : \sum_i (x_i - a_i)^2 \leq \alpha^2\}$  is called a *disc* with center  $a$  and radius  $\alpha$ , and is convex. The set  $D^0 = \{x : \sum_i (x_i - a_i)^2 < \alpha^2\}$ , is the *interior* of the disc  $D$  and is also convex. Generalization of the disc to higher dimensions is called a *hypersphere*. If  $\exists$  finite  $\alpha > 0 \ni S \subset D$ , then the set  $S$  is said to be *bounded*. Bounded convex polyhedral sets are called *polytopes*.
5. The dimension of a convex set is the dimension of the smallest (in terms of set inclusion) affine set containing it.
6.  $x \in S$  is called a *boundary point* of  $S$  if  $\forall \alpha > 0$ , the disc with center  $x$  and radius  $\alpha$  has points both inside the disc and outside it. A boundary point need not belong to the set under consideration. If  $S$  is a set that contains all its boundary points is said to be *closed*. If  $\bar{S}$  the complement of  $S$  is closed then  $S$  is said to be *open*. Another way to define an open set is to require that  $\forall$  points in the set,  $\exists \alpha > 0$  such that disc centered at the point with radius  $\alpha$  is in the set. *Please avoid the confusion between the terms "closed" and "bounded"*.
7. The set of feasible solutions to an LP is closed, convex, and polyhedral; but it is not necessarily bounded.
8. A point  $x^0 \in S$  is *not* an *extreme point (vertex)* of the convex set  $S$  if  $\exists$  points  $x^1$ , and  $x^2$  and  $\lambda \in (0, 1) \ni x^0 = \lambda x^1 + (1 - \lambda)x^2$ . Thus,  $x^0$  is an extreme point of  $S$  if for every straight line  $L$  passing through it the segment  $L \cap S$  has  $x^0$  as one end. If  $L$  is a straight line and  $S$  is a convex set  $\ni$  for every point  $\in L \cap S$ , the above statement is true for all lines  $L' \neq L$ , then we say that  $L \cap S$  is an *edge* of the set  $S$ ; if in this case,  $L \cap S$  has two end points these are extreme points of  $S$  and are said to be *adjacent*. If  $L \cap S$  is an edge which has only one end (hence is a semi-infinite line) then it is called a *ray* starting from the end and this end is an extreme point of  $S$ . If it has no end at all, then it can be shown that the set  $S$  has no extreme points at all. The disc  $D$  has an infinite number of extreme points while  $D^0$  has none; open sets do not have extreme points. A necessary and sufficient condition for a closed convex set to have extreme points is that there be no subset that is an affine set of dimension  $\geq 1$ .

9. Let  $S$  be a convex set. If  $H$  is a hyperplane  $\ni S \subset H^+$  and  $S \cap H \neq \phi$ , then  $H$  is said to be a *supporting* hyperplane of  $S$ ; if  $S$  is polyhedral then,  $S \cap H$  is called a *face* of  $S$ . Faces of the highest dimension are called *facets*.
10. A set  $S$  that satisfies  $[x \in S \implies \lambda x \in S \forall \lambda \geq 0]$  is called a *cone*. If it is convex and is a cone then it is called a *convex cone*. A convex cone that does not contain any subspace of dimension  $\geq 1$  is said to be *pointed*. If it is polyhedral set then it is called a *polyhedral cone*.

**Theorem 1** Let  $S$  be a convex set and  $x \notin \hat{S}$  where  $\hat{S}$  is the closure of  $S$  (the smallest closed set containing  $S$ ). Then,  $\exists$  a hyperplane  $H \ni \hat{S} \subset H^{++}$  and  $x \in H^{--}$ . (This is known as the separating hyperplane theorem).

Let  $x$  be a boundary point of a convex set  $S$ . Then  $\exists$  a supporting hyperplane passing through  $x$  for  $S$ .

**Theorem 2** A closed convex set is the intersection of all its supporting half-spaces.

**Theorem 3** Let  $S$  be subset of  $\mathbf{R}^n$ . Any point in the convex hull of  $S$  can be written as a convex combination of at most  $n + 1$  points in  $S$ . (This result is often called Caratheodory's theorem).

**Theorem 4** A closed bounded (compact) convex set is the convex hull of its extreme points. (This is known as Krein-Millman theorem).

**Functions:** A function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is said to be *convex* if  $\forall x^1, x^2$ , and  $\lambda \in (0, 1)$   $f(\lambda x^1 + (1 - \lambda)x^2) \leq \lambda f(x^1) + (1 - \lambda)f(x^2)$ .  $f$  is *concave* if  $-f$  is convex (or equivalently the inequality goes in the opposite direction). A function that is both convex and concave is a *linear* function. Another equivalent definition of convex functions is the following:  $f$  is convex iff  $\{(x, z) : f(x) \leq z\}$  is a convex set; this set is known in the literature as the *epigraph* of  $f$ . A simple consequence of this is the fact that the sets  $\{x : f(x) \leq \alpha\}$  are convex  $\forall \alpha$ ; these sets are known as the *level sets*. If  $f$  satisfies only the second condition then  $f$  is said to be *quasiconvex*. If the sets  $\{x : f(x) \geq \alpha\}$  are convex  $\forall \alpha$ , then  $f$  is said to be *quasiconcave*. A function that is both quasiconvex and quasiconcave is said to be *quasimonotone*. An example of such functions of a single variable is the monotone function.

## 1.1 Properties

1.  $\sum \lambda_i f^i(x)$  is convex if each  $f^i$  is convex and  $\lambda_i \geq 0 \forall i$ .
2.  $\sup_{\nu} f_{\nu}(x)$  is a convex function if each of a family of functions  $f_{\nu}$  is convex.
3. If  $f$  is convex function and  $x^0$  is a *local minimum* of  $f$  over a convex set  $S$ , then it is a *global minimum*.

4. If  $f$  is convex over  $S$ , then it is continuous in the relative interior of  $S$ .
5. If  $f$  is a differentiable function of a single variable, then  $f'$  is monotone nondecreasing is both necessary and sufficient for convexity; if  $f$  is twice differentiable then  $f'' \geq 0$  is a necessary and sufficient condition for convexity. If  $f$  is a function of several variables then these are modified to:

$$f(x) - f(x^0) \geq (x - x^0)\nabla f(x^0)$$

and the *hessian*  $H$  is *positive semidefinite*. (A matrix  $M$  is *positive semidefinite* (P.S.D)  $\iff x^t M x \geq 0 \forall x$ ).

6.  $F$  quasiconvex  $\iff f(\lambda x^1 + (1 - \lambda)x^2) \leq \max[f(x^1), f(x^2)] \forall x^1, x^2, \lambda \in [0, 1]$ .

## 1.2 LP Related Results

1. The set of feasible solutions of a LP is a convex polyhedral set.
2. Let the LP be in the form:  $[\min cx : x \geq 0; Ax = b]$ . Extreme points of the feasible set are precisely the basic feasible solutions. The simplex algorithm “*moves*” from one extreme point to an adjacent one that is “*better*” along the edge joining them. Unbounded feasible region implies the existence of nonzero solutions to the system:  $[x \geq 0; Ax = 0]$ . These are the rays of the feasible set and form a convex cone. The extreme rays correspond to the extreme points of the set  $\{x : x \geq 0; Ax = 0; \sum x_j = 1\}$ . If the LP is unbounded then  $\exists$  an extreme ray with  $cx < 0$  and this is produced by the simplex algorithm. A degenerate pivot in the simplex algorithm means that we have not “moved” at all; but are at the same solution point but with different bases. *Thus, though there is no “geometric” movement there is an algebraic pretension of such movement.*
3. We can geometrically “*show*” that if the LP has an optimal solution, then it has an extreme point that is optimal – *provided  $\exists$  extreme feasible solutions*. For this purpose, we need the following powerful result: Any point  $x^0$  in a convex closed polyhedral set  $S$  can be expressed as the sum of a convex combination of its extreme points  $P^i$  and a nonnegative combination of its extreme rays  $R^j$ . “Clearly” if the problem is bounded, then there are no “*profitable*” rays – rays with  $cx < 0$ . Hence,

$$cx^0 = \sum_i \lambda_i cP^i + \sum_j \mu_j cR^j; \lambda_i \geq 0; \sum \lambda_i = 1; \mu_j \geq 0$$

Therefore,  $cx^0 \geq \sum \lambda_i cP^i$  and hence  $\exists P^* \ni cP^* \leq cx^0$ ;  $P^*$  is the required point. *Please note that this proof can be altered to suit quasiconcave functions in general.*

4. The set of optimal solutions to an LP is a convex polyhedral set that is closed. Indeed, it is the intersection of the set of feasible solutions and a supporting hyperplane. This alone can be used to show that  $\exists$  an extreme point optimal solution if an optimal solution and an extreme point solution exist.
5. Consider the LP:

$$[\min cx : x \geq 0; Ax = b] \quad (1.1)$$

The set  $S = \{b : 1.1 \text{ is feasible}\}$  is a closed convex polyhedral cone; it is the cone generated by the columns of  $A$  and some authors denote it by  $\text{pos}(A)$ . A polyhedral cone can be defined in two ways: (i) it is generated by a finite number of generators or extreme rays; (ii) it is the intersection of a finite number of half spaces passing through the origin. The set  $T = \{b : 1.1 \text{ has an optimal solution}\}$  is also a closed convex polyhedral cone. Either or both of these cones may be empty.

6. The set of feasible solutions to an LP of the form above may be thought of as a subset of  $\mathbf{R}^{n-m}$  where  $A$  is an  $m \times n$  matrix with full row rank. In this case a nondegenerate extreme point is one where  $n - m$  hyperplanes meet; a degenerate one is where more meet.
7. The optimal value of the above LP may be thought of as a function of  $A$ ,  $b$ , and  $c$ . It is convex in  $b$  (over the feasible set of  $b$ ) for fixed  $A$  and  $c$ ; convex in  $c$  for fixed  $A$  and  $b$ . Both these are piecewise linear. One important question deals with the number of pieces for special cases.

### 1.3 Proofs

1.  $x, y \in S \cap T \implies x, y \in S$ , and hence  $\lambda x + (1 - \lambda)y \in S$ ; similarly  $\lambda x + (1 - \lambda)y \in T$  and hence it is in  $S \cap T$ . Hence the result. The assertion about  $\cap_{\alpha \in A} S_{\alpha}$  follows in a similar manner.
2. "Clearly" the assertion is true by definition for  $n = 2$ . Assume that it is true for  $n \leq k$ . Let  $x = \sum_{i=1}^k \lambda_i x^i = \lambda_1 x_1 + (1 - \lambda_1) \sum_{i=2}^k \mu_i x_i = \lambda_1 x_1 + (1 - \lambda_1)z$  where  $\mu_i = \lambda_i / (1 - \lambda_1) \geq 0$ , and  $\sum_{i=2}^{k+1} \mu_i = 1$ . (Please note that we have assumed that  $\lambda_1 > 0$ ; but this is without loss since not all  $\lambda_i$  can be 0). Since  $z$  is a convex combination of  $k$  members of  $S$ ,  $z \in S$ ; hence  $x \in S$ . Please note that this proof will not generalize to the infinite case which needs a different kind of proof.
3. To show that  $H$  is a convex set: Let  $x, y \in H$ . Hence  $cx = \alpha$ , and  $cy = \alpha$ . Thus,  $\lambda cx + (1 - \lambda)cy = c(\lambda x + (1 - \lambda)y) = \alpha$ . Hence  $H$  is convex.  $H^+$ ,  $H^{++}$ ,  $H^-$ , and  $H^{--}$  are shown to be convex in a similar manner. Indeed, this is the most primitive way of showing a set to be convex.

4. To show that  $D$  is convex: Let  $x, y \in D$ . Hence  $\sum(x_i - a_i)^2 \leq \alpha^2$ ; and  $\sum(y_i - a_i)^2 \leq \alpha^2$ . Need to show that  $\sum[\lambda(x_i + (1 - \lambda)y_i) - a_i]^2 \leq \alpha^2$ .  $\sum[\lambda(x_i + (1 - \lambda)y_i) - a_i]^2 = \sum[\lambda(x_i - a_i) + (1 - \lambda)(y_i - a_i)]^2 = \lambda^2 \sum(x_i - a_i)^2 + (1 - \lambda)^2 \sum(y_i - a_i)^2 + 2\lambda(1 - \lambda) \sum(x_i - a_i)(y_i - a_i) = \lambda \sum(x_i - a_i)^2 + (1 - \lambda) \sum(y_i - a_i)^2 - \lambda(1 - \lambda) \sum[(x_i - a_i) - (y_i - a_i)]^2 \leq \alpha^2$ .
5. The set of feasible solutions is closed because the limit points of a sequence of points in the set is in the set. The rest is easy.

### 1.3.1 Theorems

T1 Consider the problem  $\min_{y \in \hat{S}} d(x, y)$ ; note  $d(x, y) = \|x - y\|$ . Since distance is a continuous nonnegative function and  $\hat{S}$  is closed, the minimum is achieved at a point  $y^* \in \hat{S}$ , and  $y^* \neq x$ . Consider the halfspace  $H^+ = \{y : 2(y^* - x)^t y \geq (y^* - x)^t (y^* + x)\}$ . It is easy to show that  $y^* \in H^+$  and  $x \notin H^+$ . For  $y \in \hat{S}$ ,  $\lambda y + (1 - \lambda)y^* \in \hat{S}$  since  $S$  is convex. By the definition of  $y^*$ , we have:

$$\|[\lambda y + (1 - \lambda)y^*] - x\|^2 = \|y^* + \lambda(y - y^*) - x\|^2 \geq \|y^* - x\|^2$$

This on simplification yields:

$$2\lambda(y - y^*)^t (y^* - x) + \lambda^2(y - y^*)^t (y - y^*) \geq 0 \quad \forall \lambda \in [0, 1]$$

Hence:

$$(y - y^*)^t (y^* - x) \geq 0 > (x - y^*)^t (y^* - x)/2$$

and therefore,

$$2(y - y^*)^t (y^* - x) + (y^* - x)^t (y^* - x) > 0, \text{ and hence}$$

$$(2y - y^* - x)^t (y^* - x) > 0, \text{ and hence}$$

$2y^t (y^* - x) > (y^* + x)^t (y^* - x)$  which is the required result. Hence  $\hat{S} \subset H^{++}$ , and  $x \in H^{--}$ .

Since  $x$  is a boundary point of  $S$ ,  $\exists$  a sequence of points  $x_\nu \notin \hat{S} \ni \lim x_\nu = x$ . (Choose each of these to be in smaller and smaller spheres centered at  $x$  and from outside  $\hat{S}$ ). For  $x_\nu$ ,  $\exists a_\nu \ni \|a_\nu\| = 1$ , and  $a_\nu^t y > a_\nu^t x_\nu \forall y \in \hat{S}$ . Let  $a$  be a limit point of the sequence  $\{a_\nu\}$ . (Such points exist since all of these are in a compact set). Then  $a^t y \geq a^t x \forall y \in \hat{S}$  with equality holding for  $x$ . This is the required supporting plane.

??If  $S$  is a closed convex set and  $T$  is the intersection of all its supporting halfspaces, then "clearly"  $S \subseteq T$ . If  $x \in T - S$ , then by theorem 1,  $\exists H \ni S \subset H^{++}$  and  $x \notin H^{++}$ , and this is a contradiction to the definition of  $T$ .

### 1.3.2 Functions

1. Let  $f(x) = \sup f_\nu(x)$ . Since  $f_\nu(\lambda x + (1 - \lambda)y) \leq \lambda f_\nu(x) + (1 - \lambda)f_\nu(y) \forall \nu$ , the result follows.
2. Let  $N(x^0)$  be the neighborhood of  $x^0$ . If  $f(x^0) > f(x^1)$ , then  $\exists z = \lambda x^0 + (1 - \lambda)x^1$  and  $z \in N(x^0)$  with  $f(z) < f(x^0)$ .

## 1.4 LP Related Results

1. Let  $x$  be a basic feasible solution corresponding to the basis  $B$  (think of a basis as a set of linearly independent columns of  $A$ ). By this we mean that  $x^{NB} = 0$ , and  $Bx^B = b$  has a unique solution. If  $x = \lambda y + (1 - \lambda)z$ , then  $y^{NB} = z^{NB} = 0$ ; hence  $y^B = z^B = x^B$ . Hence  $x = y = z$ . Hence  $x$  is an extreme point.

Conversely, suppose  $x$  is an extreme point, and let  $B$  be the columns corresponding to positive components of  $x$ . Suppose these are dependent;  $\exists z^B \neq 0 \ni Bz^B = 0$ . By letting other components equal to 0 we get  $z \neq 0 \ni Az = 0$ . Consider the vector  $x + \theta z$ . Since  $x^{NB} = z^{NB} = 0$ ;  $(x + \theta z)^{NB} = 0 \forall \theta$ ; also since  $Az = 0$ ;  $A(x + \theta z) = b \forall \theta$ . Since  $x^B > 0$ ,  $\exists \theta > 0 \ni (x + \theta z)^B \geq 0$ . This is a contradiction of extremeness of  $x$ . Hence the columns of  $B$  are independent and hence  $x$  is a basic feasible solution.