Solution to Assignment #2:

1. (4.3-6) Show that $t(n) = O(n \lg n)$ for the relation:

$$t(n) = 2t\left(\left\lfloor \frac{n}{2} \right\rfloor + 17\right) + n$$

**Solution:** Let $m = n - 34$. This implies that $\left\lfloor \frac{n}{2} \right\rfloor + 17 = \left\lfloor \frac{m}{2} \right\rfloor + 34$. Hence our equation becomes:

$$t(m + 34) = 2t\left(\left\lfloor \frac{m}{2} \right\rfloor + 34\right) + (m + 34)$$

Now let $s(m) = t(m + 34)$ and the equations changes to

$$s(m) = 2s\left(\left\lfloor \frac{m}{2} \right\rfloor \right) + \Theta(m)$$

$$= \Theta(m \lg m)$$

if we let $S(m) = t(m + 34)$. Hence,

$$t(n) = t(m + 34) = s(m) = \Theta(m \lg m) = \Theta((n - 34) \lg(n - 34))$$

(a) Now we show that $(n + c) \lg(n + c) = \Theta(n \lg n)$ and this will complete the proof.

$$\lim_{n \to \infty} \frac{(n + c) \lg(n + c) - n \lg n}{n \lg n}$$

$$= \lim_{n \to \infty} \left(\frac{\lg(n + c) - \lg n}{\lg n}\right) + \lim_{n \to \infty} \frac{c \lg(n + c)}{n \lg n}$$

$$= \lim_{n \to \infty} \left(\frac{\lg(1 + \frac{c}{n})}{\lg n}\right) = 0$$

Hence

$$\lim_{n \to \infty} \frac{(n + c) \lg(n + c)}{n \lg n} = 1$$

and therefore, $(n + c) \lg(n + c) = \Theta(n \lg n)$.

(b) 4.3-9: Solve the relation $t(n) = 3t(\sqrt{n}) + \log n$

**Solution:**

Let $\log n = m; n = 2^m$. The above equation changes to

$$t(2^m) = 3t(2^{\frac{m}{2}}) + m$$

Now let $t(2^x) = s(x)$ and the equations changes to

$$s(m) = 3s\left(\frac{m}{2}\right) + m$$

This equation is solved by master theorem (case 1) to yield the solution $s(m) = \Theta(m^{\log_2 3})$. Hence the solution of the original equation is $t(n) = \Theta((\log n)^{\log_2 3})$. 

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2. 4.4-2: Use a recursion tree to give an asymptotically tight solution to the relation:

\[ t(n) = t\left(\frac{n}{2}\right) + n^2 \]

Use substitution method to verify your answer.

\[
\begin{align*}
&n^2 \\
&\langle (n/2)^2 \rangle \\
&\langle (n/4)^2 \rangle \\
&\langle (n/2^k)^2 \rangle
\end{align*}
\]

where \( k = \log n \). Adding we get

\[
t(n) = n^2 \sum_{j=0}^{k} \left(\frac{1}{2}\right)^j = \Theta(n^2).
\]

**Verification:** By using induction hypothesis on smaller values we get

\[
t(n) \leq n^2 + c\left(\frac{n}{2}\right)^2 = n^2[1 + \frac{c}{4}] \leq cn^2 \text{if } c > \frac{4}{3}
\]

This shows that \( t(n) = O(n^2) \). By a similar analysis you can show that \( t(n) = \Omega(n^2) \).

4.4-6: Use a recursion tree to give an asymptotically tight solution to the relation:

\[ t(n) = t(\alpha n) + t((1-\alpha)n) + cn \]
where $c > 0$, and $0 < \alpha < 1$ are given constants.

The depth of the tree is $\frac{\log n}{\log \alpha} = \Theta(\log n)$. Hence,

$$t(n) = \Theta(n \log n)$$

Hence $t(n) = \Omega(n \log n)$.

3. **Problem 4-3**: (b),(c),(f)

   a) $t(n) = 3t\left(\frac{n}{3}\right) + \frac{n}{\log n}$

   $f(n) = \frac{n}{\log n}; n^{\log_3 a} = n$. So both cases 2 and 3 do not apply. Moreover, $\lim_{n \to \infty} \left[\frac{n^{1-\epsilon}}{f(n)}\right] = 0$ and hence $f(n) = \omega(n^{1-\epsilon})$. So we have $f(n) \neq O(n^{\log_3 a-\epsilon})$. So the master theorem does not apply in this case. This equation is of the form $t(n) = at\left(\frac{n}{b}\right) + n^{\log_b a}(\log n)^k; b > 1; a \geq 1$ but $k \geq 0$. In this case $k = -1$. So we try iteration method after changing variables.

   Let $\log_3 n = k; n = 3^k; t(3^k) = s(k)$ and the equation becomes:

   $$t(n) = s(k) = 3s(k - 1) + \frac{3^k}{c^k}$$

   $$= \frac{3^k}{c^k} + 3^k \frac{3^{k-1}}{c(k-1)} + \ldots$$

   $$= \frac{1}{c} \sum_{i=1}^{k} \frac{1}{i}$$

   $$= \Theta(3^k \log k)$$

   $$= \Theta(n \log \log n)$$

   Here $\log n = \log_3 n, c = \frac{1}{\log_3 n}.$
c) $t(n) = 4t(n/2) + n^2\sqrt{n}$

$$f(n) = n^{2.5} = \Omega(n^2); af(n) = 4(n^{2.5}) = \frac{2^2}{\sqrt{2}} \leq 0.8n^{2.5}$$

Hence this is case 3 of master theorem and so the result is $t(n) = \Theta(n^{2.5})$.

f) $t(n) = t(n/4) + t(n/4) + t(n/4) + n$

Substitution Method: Since $1/2 + 1/4 + 1/8 = \frac{7}{8} < 1$, guess $t(n) \leq cn$ and prove by induction. Using induction hypothesis

$$t(n) \leq \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)cn + n$$

$$= \frac{7}{8}cn + n$$

$$\leq cn \text{ for } c \geq 8$$

This completes the proof.

4. (4-4) Fibonacci Sequence:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad i \geq 2$$

(a)

$$\mathcal{F}(z) = \sum_{i=0}^{\infty} F_i z^i = z + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2})z^i$$

$$= z + z \sum_{i=0}^{\infty} F_i z^i + z^2 \sum_{i=0}^{\infty} F_i z^i$$

$$= z + z\mathcal{F}(z) + z^2\mathcal{F}(z)$$

$$= \frac{z}{1 - z - z^2}$$

$$= \frac{z}{(1 - \phi z)(1 - \hat{\phi} z)}$$

the last of these follows from the fact that $\frac{1}{\phi}$ and $\frac{1}{\hat{\phi}}$ are the roots of $[1 - z - z^2 = 0]$.

(b)

$$\frac{z}{(1 - \phi z)(1 - \hat{\phi} z)} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi z} - \frac{1}{1 - \hat{\phi} z}\right)$$

$$\frac{1}{(1 - \phi z)} = \sum_{i=0}^{\infty} \phi^i z^i \text{ and } \frac{1}{(1 - \hat{\phi} z)} = \sum_{i=0}^{\infty} \hat{\phi}^i z^i \text{. Hence}$$

(c)

$$\mathcal{F}(z) = \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} (\phi^i - \hat{\phi}^i) z^i$$
(d) Hence

\[ F_i = \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i) \]

and \( |\hat{\phi}^i| < 1 \). Hence \( F_i = \left[ \frac{\phi^i}{\sqrt{5}} \right] \).

(e) Want to show that \( F_{i+2} \geq \phi^i \) for \( i \geq 0 \)

We do this by induction. Clearly true for \( i = 0 \). Suppose it is true for \( i \leq k \) Will show for \( i = k + 1 \).

\[ F_{k+3} = F_{k+1} + F_{k+2} \geq \phi^{k-1} + \phi^k = \phi^{k+1} \left[ \frac{1}{\phi^2} + \frac{1}{\phi} \right] = \phi^{k+1} \left[ \frac{\phi + 1}{\phi^2} \right] = \phi^{k+1}. \]

\[
\frac{1}{\phi^2} = 1 - \frac{1}{\phi} \\
\phi^2 = 1 + \phi \\
\left[ \frac{\phi + 1}{\phi^2} \right] = 1
\]

(Recall \( \frac{1}{\phi} \) is a solution of the equation

\[ 1 - z - z^2 = 0 \]

The result now follows.)