Chapter 16

Partial differential equations

16.1 Introduction

Partial differential equations (PDEs) play a major role in electromagnetic engineering and mechanical engineering, as well as most fields of physics. Methods for solving PDEs depend on the order of the PDE (the order of the highest partial derivative) and, for PDEs of order 2 or greater, on the type of the PDE. We discuss first-order PDEs, second-order PDEs, and then a few essential aspects of numerical methods.

An example of a second-order PDE that is of considerable importance in modern communications technology is the beam-propagation equation

$$\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} + \frac{i}{2}\beta_2\frac{\partial^2 u}{\partial t^2} = -\frac{\alpha}{2}u + i\gamma|u|^2 u, \tag{16.1}$$

which describes the propagation of a guided optical wave with an electric-field intensity that is proportional to u along a fiber that extends in the z direction. In this equation, the term $-\alpha u/2$ describes linear attenuation, the term $\beta_2\partial^2 u/\partial t^2$ describes chromatic dispersion, which broadens pulses in time, and the term $\gamma|u|^2 u$ describes some of the nonlinear-optical properties of the fiber such as self-phase modulation, cross-phase modulation and four-wave mixing. The design of a fiber transmission system typically involves many numerical simulations using the beam-propagation equation.

An ordinary differential equation such as

$$\frac{dz}{dt} = f(z,t) \tag{16.2}$$

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determines a family of curves $z(t)$, from which the initial condition or a boundary condition selects one curve as the solution. However, a partial differential equation in two independent variables z, t determines a family of surfaces $u(z,t)$, from which an initial condition, a set of boundary conditions, or a combination of initial and boundary conditions selects one surface as the solution.

There are three different types of second-order PDEs, called hyperbolic, parabolic and elliptic. PDEs of hyperbolic type describe initial- and boundary-value problems involving wave motion; PDEs of parabolic type describe either initial-value diffusion processes or one-way wave motion; and PDEs of elliptic type describe boundary-value problems such as occur in electrostatics. Each type of second-order PDE requires its own set of numerical methods. Methods for hyperbolic equations will not work for elliptic equations, for example.

16.2 First-order, quasilinear PDEs

16.2.1 Basic definitions

A first-order partial differential equation of the form

$$\mathbf{a}(u,\mathbf{r}) \cdot \nabla u(\mathbf{r}) = c(u,\mathbf{r}) \tag{16.3}$$

(where $\mathbf{r} \in \mathbb{R}^n$ and u is the unknown function to be solved for) is called **quasilinear**. If the source function c is the null function, $c \equiv 0$, then the equation is called **homogeneous**. If c is not the null function, then the equation is **inhomogeneous**. Methods for the homogeneous and inhomogeneous cases differ in some essential respects.

We shall use the beam-propagation equation with dispersion and linear attenuation turned off,

$$\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} = i\gamma |u|^2 u, \tag{16.4}$$

to illustrate both analytical and numerical methods for first-order quasilinear PDEs. In this example,

$$a = 1, \quad b = \frac{1}{v}, \quad c = i\gamma |u|^2 u. \tag{16.5}$$

If we also set $\gamma = 0$, then the beam-propagation is homogeneous; otherwise, it is inhomogeneous.

The solutions of a homogeneous quasilinear partial differential equation do not obey the superposition principle if the coefficient vector $\mathbf{a}$ depends nontrivially on the solution u. For example, if u_1 and u_2 are both solutions of **Burgers' equation**,

$$u\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = 0, \tag{16.6}$$

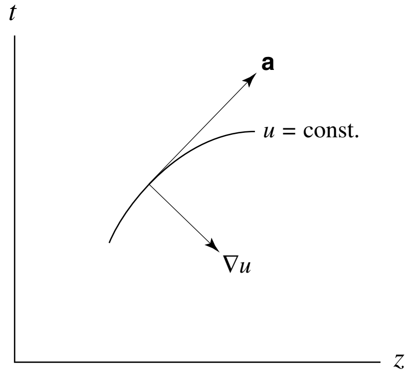

Figure 16.1: The vector $\mathbf{a}(u,\mathbf{r})$ is orthogonal to ∇u, and is tangent to the curve $u =$ constant.

then αu_1 is not a solution unless $\alpha = 1$, because (unless $u_1 \equiv 0$)

$$\alpha^2 u_1 \frac{\partial u_1}{\partial z} + \alpha \frac{\partial u_1}{\partial t} = (\alpha^2 - 1) u_1 \frac{\partial u_1}{\partial z} + (\alpha - 1)\frac{\partial u_1}{\partial t} \neq 0, \tag{16.7}$$

and $u_1 + u_2$ is not a solution unless $u_1 \equiv 0$ or $u_2 \equiv 0$ (or both), because

$$(u_1 + u_2)\frac{\partial (u_1 + u_2)}{\partial z} + \frac{\partial (u_1 + u_2)}{\partial t} = u_1 \frac{\partial u_2}{\partial z} + u_2 \frac{\partial u_1}{\partial z} \neq 0. \tag{16.8}$$

The inapplicability of the superposition principle makes the theory of quasilinear PDEs less than straightforward, because one cannot appeal for help from linear algebra.

16.2.2 First-order, quasilinear, homogeneous PDEs

16.2.2.1 Introduction and motivation

The homogeneous equation

$$\mathbf{a}(u,\mathbf{r}) \cdot \nabla u(\mathbf{r}) = 0 \tag{16.9}$$

implies that the vector $\mathbf{a}(u,\mathbf{r})$ is orthogonal to ∇u at every point $\mathbf{r}$ of the domain of u (Fig. 16.1). Since ∇u is orthogonal to the level surfaces of u, the vector $\mathbf{a}$ is tangent to the level surface

$$u(\mathbf{r}) = \text{constant} \tag{16.10}$$

at every $\mathbf{r} \in \text{domain}\,[u]$. Thus the equation $\mathbf{a} \cdot \nabla u = 0$ defines the level surfaces of u by defining a family of tangent lines or planes. We show how to construct the function u by constructing its level surfaces.

16.2.2.2 Characteristic curves in the two-dimensional case

We specialize now to the two-dimensional case, in which Eq. (16.9) reads

$$a(u,z,t)\frac{\partial u}{\partial z}+b(u,z,t)\frac{\partial u}{\partial t}=0 \tag{16.11}$$

where

$$z=x^1,\quad t=x^2,\quad a=a^1,\quad b=a^2. \tag{16.12}$$

This equation implies that the vector field

$$\mathbf{a}(u,z,t)=\begin{pmatrix}a(u,z,t)\\ b(u,z,t)\end{pmatrix} \tag{16.13}$$

is tangent to the curve in the x–t plane that is defined by the condition $u(z,t)=$ constant. It is equivalent to say that the derivative of u in a direction that is parallel to $\mathbf{a}$ is zero, because the directional derivative of u in the direction that is specified by the unit vector $\hat{\mathbf{s}}$ is $\hat{\mathbf{s}}\cdot\nabla u$.

For example, Burgers' equation (16.6) is a homogeneous quasilinear equation of the form (16.11) with $a(u,z,t)=u$ and $b(u,z,t)=1$. The left-hand side of Burgers' equation is a special case of the convective derivative of the velocity, $\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}$, which occurs in the Navier-Stokes equations of fluid dynamics.

Let C be a rectifiable curve in the z–t plane, and let

$$z=z(s),\quad t=t(s) \tag{16.14}$$

be the parametric equations of C in terms of the arc length s. The derivative of u along C is

$$\frac{du}{ds}=\frac{\partial u}{\partial z}\frac{dz}{ds}+\frac{\partial u}{\partial t}\frac{dt}{ds}. \tag{16.15}$$

If C is a curve on which the function u has a constant value, then $du/ds=0$ on C. Then Eq. (16.15) implies that the vector

$$\mathbf{t}(u,z,t)=\begin{pmatrix}\dfrac{dz}{ds}\\[2ex] \dfrac{dt}{ds}\end{pmatrix} \tag{16.16}$$

is tangent to C.

We now have two vector fields, $\mathbf{a}$ $\big($Eq. (16.13)$\big)$ and $\mathbf{t}$ $\big($Eq. (16.16)$\big)$, both of which are tangent to the curve C, on which u has a constant value. It follows that $\mathbf{a}$ and $\mathbf{t}$ are parallel. Therefore, for each value of arc length (s), $\mathbf{t}$ is proportional to $\mathbf{a}$:

$$\mathbf{t}(s)=F(s)\,\mathbf{a}(s). \tag{16.17}$$

It follows that

$$\begin{aligned}\frac{dz}{ds} &= F(s)\,a\Big(u\big(z(s),t(s)\big),z(s),t(s)\Big),\\ \frac{dt}{ds} &= F(s)\,b\Big(u\big(z(s),t(s)\big),z(s),t(s)\Big).\end{aligned} \tag{16.18}$$

For a rectifiable curve in $\mathbb{R}^2$, the tangent vector $\mathbf{t}$ is a unit vector:

$$\mathbf{t}\cdot\mathbf{t} = \left(\frac{dz}{ds}\right)^2 + \left(\frac{dt}{ds}\right)^2 = 1. \tag{16.19}$$

It follows that dz/ds and dt/ds cannot vanish simultaneously. As long as a and b are finite and nonzero, it follows that $F(s) \neq 0$. The next step is to show how to use Eq. (16.18) to determine the curve C and the solution, u.

Dividing the second equation in (16.18) by the first equation, one obtains the ordinary differential equation

$$\boxed{\frac{dt}{dz} = \frac{dt/ds}{dz/ds} = \frac{b}{a}} \tag{16.20}$$

which determines the curve C. A curve determined by this differential equation is called a **characteristic curve**.

16.2.2.3 The PDE along a characteristic curve

From Eqs. (16.15) and (16.18) one has

$$\frac{du}{ds} = F(s)\left(a\frac{\partial u}{\partial z} + b\frac{\partial u}{\partial t}\right). \tag{16.21}$$

From this equation and the original partial differential equation (16.11) one sees that, on a characteristic curve,

$$\boxed{\frac{du}{ds} = F(s)\,c\Big(u\big(z(s),t(s)\big),z(s),t(s)\Big).} \tag{16.22}$$

In the homogeneous case ($c = 0$), u is constant along a characteristic curve $\big(du/ds = 0\big)$.

Eq. (16.22), which describes the rate of change of u as one goes along the characteristic curve, replaces the original PDE, Eq. (16.11). Since differentiation with respect to s may not be convenient, we re-express du/ds in terms of a derivative with respect to z along the characteristic curve, in order to have a more useful expression than (16.22) for the inhomogeneous case. From the relation

$$F = a^{-1}dz/ds, \tag{16.23}$$

one has

$$\frac{1}{F}\frac{du}{ds} = a\left(\frac{du}{dz}\right)_C \tag{16.24}$$

where $(du/dz)_C$ is the rate of change of u as z is varied along the characteristic curve C. For regions in which a does not vanish, one can therefore replace the original PDE or Eq. (16.22) with the **characteristic differential equation**

$$\boxed{\left(\frac{du}{dz}\right)_C = \frac{\partial u}{\partial z} + \frac{b}{a}\frac{\partial u}{\partial t} = \frac{c}{a}.} \tag{16.25}$$

The subscript C indicates that the derivative with respect to z is evaluated along the characteristic curve C, according to Eq. (16.24). Again, in the homogeneous case $(c = 0)$, $(du/dz)_C = 0$.

Conversely, let z obey the ordinary differential equation $dt/dz = b/a$ [Eq. (16.20)] and let C be the curve determined by this differential equation. It follows that Eq. (16.21) holds along C. Let s be the arc length along C, and let u be any adequately differentiable function that is constant along C. Then $du/ds = 0$. By the chain rule (16.15),

$$\begin{aligned}\frac{du}{ds} &= \frac{\partial u}{\partial z}\frac{dz}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} \\ &= Fa\frac{\partial u}{\partial z} + Fb\frac{\partial u}{\partial t}.\end{aligned} \tag{16.26}$$

Since $F(s) \neq 0$, this equation and the equation $du/ds = 0$ imply Eq. (16.11). Therefore a function u satisfies the homogeneous equation (16.11) if and only if u is constant along the characteristic curve defined by Eq. (16.20).

It follows that the general solution of the two-dimensional, homogeneous, quasilinear partial differential equation (16.11) is

$$\boxed{\begin{aligned} u &= k = \text{constant} \\ \frac{dt}{dz} &= \frac{b(k,z,t)}{a(k,z,t)}.\end{aligned}} \tag{16.27}$$

The second equation determines a characteristic curve C along which $u = k$.

16.2.2.4 Initial conditions

Because the original PDE, Eq. (16.3), is equivalent to an ordinary differential equation along a characteristic curve, Eq. (16.3), it is not possible to prescribe the dependence of u along a characteristic curve.

Obviously the characteristic differential equation (16.25) does not determine u uniquely, because the characteristic differential equation determines a family of solutions u. In order to determine a unique solution of the characteristic differential equation (16.25) on a specific characteristic curve, one must prescribe an initial value of u at some point on the characteristic curve.

Let N be any curve such that the slope of N at any point (z, t) is not equal to the slope of the characteristic curve through (z, t). Prescribing the initial values of u on N is enough to determine a unique solution of the original PDE (16.3) on each characteristic curve that issues from a point on N. Therefore the values of u on N uniquely determine u, and conversely.

For simplicity, one usually requires the non-characteristic curve on which one prescribes the initial data to be smooth (*i.e.*, one chooses the parametric equations that determine the curve to have derivatives of all orders).

16.2.2.5 The linear, homogenous, dispersionless beam-propagation equation

The first-order, linear, homogeneous beam-propagation equation (without attenuation or dispersion) is

$$\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} = 0. \tag{16.28}$$

From Eq. (16.5), which gives the coefficients a and b, the ordinary differential equation that determines the characteristic curves is

$$\frac{dt}{dz} = \frac{1}{v}. \tag{16.29}$$

Since v is a constant in this example, the characteristic curves for the beam-propagation problem are straight lines in the z–t plane, with a slope equal to $1/v$,

$$z - vt = \text{constant} = z_0, \tag{16.30}$$

as illustrated in Fig. 16.2.

From Eq. (16.27), the solution of the first-order, homogeneous beam-propagation equation (without attenuation) is

$$u = k = \text{constant} \tag{16.31}$$

on a characteristic curve. This means that the value of u is uniquely determined by the value of z_0, *i.e.*, that

$$u(z, t) = f(z - vt), \tag{16.32}$$

where $f \in \mathcal{C}^1\big((-\infty, \infty); \mathbb{R}\big)$. This equation describes a pulse or wave that moves with a constant velocity v and with a waveform determined by the function f.

In order to determine u uniquely, one must specify the function f. The line $t = 0$ is a convenient non-characteristic curve, on which

$$u = f(z). \tag{16.33}$$

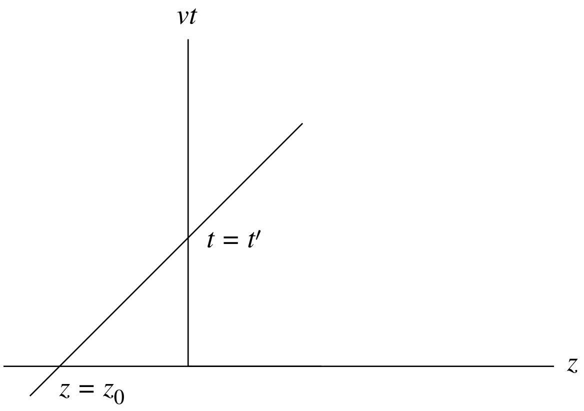

Figure 16.2: The characteristic $z - vt = z_0$ of the beam-propagation equation intercepts the z-axis at z_0 and the t-axis at $t' = -z_0/v$.

On the characteristic curve that is specified by z_0, the (constant) value of u is $f(z_0)$.

In this example, the function F in Eq. (16.17) is just a constant. From Eq. (16.23) and Fig. 16.2,

$$ds = \sqrt{2}\,dz \Rightarrow F(s) = \frac{1}{\sqrt{2}}. \tag{16.34}$$

The simplicity of this relation is a result of the fact that the characteristic curves in this example are straight lines.

16.2.2.6 Co-moving coordinates

Regarding u as a function of z_0, which is the z-intercept (the value of z at $t = 0$) for a particular characteristic line amounts to taking a "snapshot" of the wave represented by u at time $t = 0$, and then using the snapshot as an initial condition for u. Usually it is more convenient to regard u as a function of the t-intercept, t' (the value of t for which $z = 0$ for a particular characteristic), because $u(t')$ is the waveform that is "transmitted" from the point $z = 0$. Fig. 16.2 illustrates that, because t' is the t-intercept of the line $z - vt = z_0$, it follows that

$$-vt' = z_0. \tag{16.35}$$

Therefore

$$t' = -\frac{z_0}{v} = -\frac{z - vt}{v} = t - \frac{z}{v}. \tag{16.36}$$

Physically, t' is time referred to a new origin that depends on z. The relation $t' = 0$ defines t as the time at which a short pulse that starts at $z = 0$ at $t = 0$

arrives at z. The equation

$$u = g(t') \tag{16.37}$$

describes the solution of the beam-propagation equation (without attenuation, optical nonlinearities or dispersion) in terms of the time-dependent waveform, g, observed at a particular location, z.

For consistency with the point of view that t' is time measured in a new coordinate system, we define the transformed z coordinate as $z' = z$. The equations

$$\begin{aligned} z' &= z, \\ t' &= t - \frac{z}{v} \end{aligned} \tag{16.38}$$

define the new coordinates in terms of the old. Because the value of z for which $t' = t_0$ (where t_0 is arbitrary) increases linearly in t with velocity v,

$$z = v(t - t_0), \tag{16.39}$$

one often refers to the coordinates defined in Eq. (16.38) as **co-moving coordinates**.

Eq. (16.25), which replaces the original homogeneous PDE, becomes

$$\boxed{\left(\frac{du}{dz'}\right)_C = \frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} = 0.} \tag{16.40}$$

From Eq. (16.37), the general solution of this equation is (again)

$$u(z,t) = f(z - vt) = f(-vt') \tag{16.41}$$

where f is any adequately differentiable function of one variable.

Often co-moving coordinates are derived using the chain rule for partial differentiation, without explicit reference to the method of characteristics. From Eq. (16.38) and the chain rule, one finds that

$$\frac{\partial u}{\partial z'} = \frac{\partial z}{\partial z'}\frac{\partial u}{\partial z} + \frac{\partial t}{\partial z'}\frac{\partial u}{\partial t} = \frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} \tag{16.42}$$

and

$$\frac{\partial u}{\partial t'} = \frac{\partial u}{\partial t}. \tag{16.43}$$

In this approach, the original beam-propagation Eq. (16.1) becomes

$$\frac{\partial u}{\partial z'} + \frac{i}{2}\beta_2\frac{\partial^2 u}{\partial t'^2} = -\frac{\alpha}{2}u + i\gamma|u|^2 u, \tag{16.44}$$

Collecting terms, one gets the result that

$$\frac{\partial u}{\partial z} = \frac{f'}{1+f't}, \tag{16.51}$$

where f' is the derivative of f with respect to its argument. One shows similarly that

$$\frac{\partial u}{\partial t} = -\frac{f'u}{1+f't}. \tag{16.52}$$

It follows immediately that $u\partial u/\partial z + \partial u/\partial t = 0$.

We show now that u can develop a discontinuity even if the initial-value function f is continuous. The basic idea is simple: If the wave velocity is u, then an initial value of u that is greater on the left side of a particular z than on the right side of the same z will create waves that travel faster on the left side of z than on the right side. The fast waves will overtake the slow waves, causing a discontinuity.

For example, let

$$f(z) = \begin{cases} 1, & \text{if } z \le 0; \\ 1-z, & \text{if } 0 \le z \le 1; \\ 0, & \text{if } z \ge 1. \end{cases} \tag{16.53}$$

In this case, the initial velocity is 1 for negative z and 0 for $z \ge 1$ (see Fig. 16.3). The result that u is constant on characteristic curves that issue from the z-axis implies that the characteristic curves look as shown in Fig. 16.4.

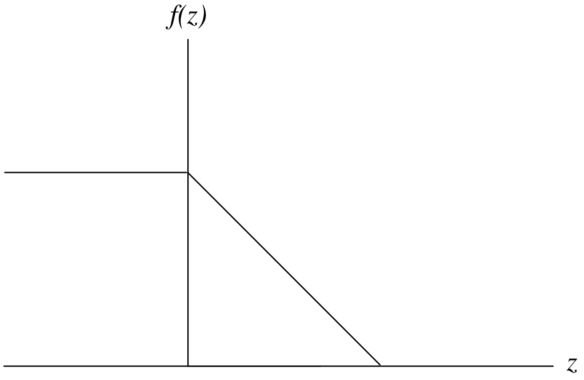

Figure 16.3: The initial velocity for Burgers' equation.

From Fig. 16.4, it is clear that the waves that issue from the negative real axis overtake the waves that issue from the interval $(1,\infty)$ at a finite time. The function u is discontinuous at a point at which two characteristic curves intersect. To the left of the red line in Fig. 16.4, $u = 1$; to the right of the red line, $u = 0$. The red line itself is a **shock front**, which propagates at the velocity $v_{\text{shock}} = \frac{1}{2}$ (which is not equal to the velocity of wave propagation). Across the shock front, u is discontinuous.

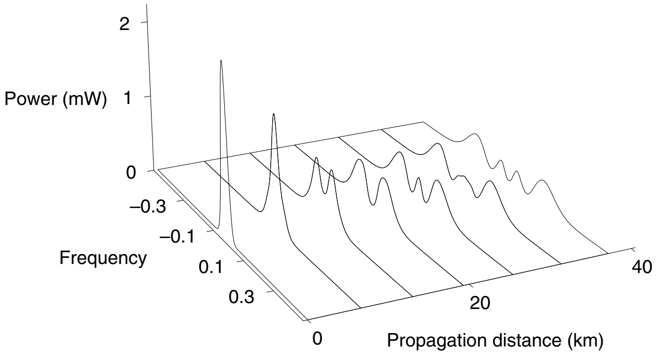

Figure 16.5: Fourier transform of $u(z', t')$ with respect to t' for selected values of z'. Courtesy of D. M. Hollenbeck.

1024. The computed function u shows substantial spectral broadening due to self-phase modulation.

Eqs. (16.27) and (16.54) determine u as a function defined on a characteristic curve and not as a function defined on $\mathbb{R}^2$. In order to compute the value $u(z, t)$ for given values of z and t for use in other applications such as the discrete Fourier transform, one must first compute u along a finite set of characteristic curves as a function of z' and t', and then, if necessary, approximate $u(z, t)$ by interpolating between points on different characteristic curves.

Exercises for Section 16.2

16.2.1 The purpose of this exercise is to show that the attenuation term in Eq. (16.45) can be removed by introducing an integrating factor. Let

$$u(z', t') = e^{-\alpha z'/2} y(z', t'), \tag{16.58}$$

and show that y satisfies the partial differential equation

$$\frac{\partial y}{\partial z'} = i\gamma e^{-\alpha z'} |y|^2 y. \tag{16.59}$$

16.2.2 Show that, if u satisfies Eq. (16.56), then $|u|^2$ is constant on a characteristic curve. (The value of the constant depends on the characteristic curve.) Deduce the result that the nonlinear right-hand side of Eq. (16.56) does not alter the power waveform in time, which is proportional to $|u(z', t')|^2$, considered as a function of t'.

16.2.3 From Eq. (16.59), show that

$$|y(z', t')|^2 = |y(0, t')|^2 \tag{16.60}$$

and therefore that

$$\begin{aligned} y(z', t') &= y(0, t') \exp\left[i\gamma |y(0, t')|^2 \int_0^{z'} e^{-\alpha z''} dz'' \right] \\ &= y(0, t') \exp\left[i\gamma |y(0, t')|^2 L_e(z') \right], \end{aligned} \tag{16.61}$$

where

$$L_e(z') = \frac{1 - \exp(-\alpha z')}{\alpha} \tag{16.62}$$

is often called the **effective length** of the nonlinear interaction.

16.2.4 Let u be the solution of Burgers' equation obtained in Section 16.2.2.7. Plot $u(z = 1.5, t)$ as a function of t from $t = 0$ to $t = 4$.

16.3 Classification of second-order, quasilinear PDEs

16.3.1 Definitions and motivation

The most general second-order quasilinear PDE in two independent variables is

$$\begin{aligned} a(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x^2} + b(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x \partial y} \\ + c(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial y^2} + e(x,y,u,u_x,u_y) = 0 \end{aligned} \tag{16.63}$$

where $u_x = \partial u/\partial x$, etc. Solving such an equation means finding a numerical or analytic function u (and its derivatives) as functions of x and y in the plane, given values of u (and a derivative of u that we will specify below) on a curve Γ. In order to discover what data we need along Γ, let us assume that we know u (and all of its derivative) on Γ, let s be the arc length along Γ, and let

$$\mathbf{s} = \begin{pmatrix} \dfrac{dx}{ds} \\ \dfrac{dy}{ds} \end{pmatrix} \tag{16.64}$$

be the tangent vector to Γ at s. As is true for $\mathbf{t}$, Eq. (16.19), $\mathbf{s}$ is a unit vector. Since we know u along Γ, we can compute the derivative of u along Γ,

$$\frac{du}{ds} = \mathbf{s} \cdot \nabla u = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}. \tag{16.65}$$

In this equation, we know the values of dx/ds and dy/ds, because we have assumed that we know how to define Γ. However, we do not know the values of the first partial derivatives of u. If we knew the values of $\partial u/\partial x$ and $\partial u/\partial y$, then we could find the value of u at a point $(x + \Delta x, y + \Delta y)$ that is close to a point (x, y) on Γ by using the approximation

$$u(x + \Delta x, y + \Delta y) \approx u(x, y) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y. \tag{16.66}$$

Our first goal, then, is to compute $\partial u/\partial x$ and $\partial u/\partial y$ given data along Γ such as u. Of course, the value of the function u and the values of its first partial derivatives are not enough to compute u at points that are far from Γ. To do that, it is enough to know *all* of the partial derivatives of u, because that permits us to compute u via a power series expansion:

$$u(x', y') = u(x, y) + \frac{\partial u}{\partial x}(x' - x) + \frac{\partial u}{\partial y}(y' - y) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x' - x)^2 + \cdots \tag{16.67}$$

In this equation, (x, y) is a point on Γ, and (x', y') is a point in the plane, but not on Γ.

16.3.2 Determination of the first partial derivatives of u

It turns out that, if we also know the normal derivative, du/dn, of u along Γ, then we can compute the first partial derivatives of u. The unit vector that is normal to Γ at s is

$$\mathbf{n} = \begin{pmatrix} -\dfrac{dy}{ds} \\ \dfrac{dx}{ds} \end{pmatrix}. \tag{16.68}$$

(Reader: Please verify that $\mathbf{n} \cdot \mathbf{s} = 0$ before reading further!) Therefore the normal derivative of u at s is

$$\frac{du}{dn} = \mathbf{n} \cdot \nabla u = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds}. \tag{16.69}$$

The system of linear equations

$$\begin{aligned} \frac{du}{ds} &= \frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dy}{ds}\frac{\partial u}{\partial y} \\ \frac{du}{dn} &= -\frac{dy}{ds}\frac{\partial u}{\partial x} + \frac{dx}{ds}\frac{\partial u}{\partial y} \end{aligned} \tag{16.70}$$

can be solved for the unknowns $\partial u/\partial x$ and $\partial u/\partial y$ provided that the determinant of coefficients is non-zero. The determinant of the coefficients of $\partial u/\partial x$ and $\partial u/\partial y$ is

$$\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ -\dfrac{dy}{ds} & \dfrac{dx}{ds} \end{vmatrix} = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1, \tag{16.71}$$

which is certainly non-zero. Therefore the unique solution of the linear system (16.70) is

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} \dfrac{du}{ds} & \dfrac{dy}{ds} \\ \dfrac{du}{dn} & \dfrac{dx}{ds} \end{vmatrix}}{\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ -\dfrac{dy}{ds} & \dfrac{dx}{ds} \end{vmatrix}} = \frac{du}{ds}\frac{dx}{ds} - \frac{du}{dn}\frac{dy}{ds} \tag{16.72}$$

and

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{du}{ds} \\ -\dfrac{dy}{ds} & \dfrac{du}{dn} \end{vmatrix}}{\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ -\dfrac{dy}{ds} & \dfrac{dx}{ds} \end{vmatrix}} = \frac{du}{ds}\frac{dy}{ds} + \frac{du}{dn}\frac{dx}{ds}. \tag{16.73}$$

Now that we have succeeded in computing the first partial derivatives given the values of u and du/dn on Γ, we will try to compute partial derivatives of higher order.

16.3.3 Boundary and initial conditions

The n independent variables (x, y, etc.) in a partial differential equation define points $(x, y, ...)$ in $\mathbb{R}^n$. It is convenient to distinguish **boundary conditions**, which are specified on the boundary of a bounded, closed subset of $\mathbb{R}^n$ (Volume 1, p. 548), from **initial conditions**, which are specified on a surface or curve that divides $\mathbb{R}^n$ into two disjoint unbounded open subsets. For example, boundary conditions can be specified on the surface of a sphere of radius a in $\mathbb{R}^n$, while initial conditions can be specified on a surface such as $x^0 = 0$ that cuts $\mathbb{R}^n$ into two disjoint subsets, $x^0 < 0$ and $x^0 > 0$, each of which is open and extends to infinity.

Physically, initial conditions are most useful for wave propagation, while boundary conditions are most useful for time-stationary or single-frequency problems such as Laplace's equation or Helmholtz's equation. Boundary conditions at infinity, which one encounters often in scattering theory, are a limiting case of boundary conditions according to the definition given above.

Historically, one of the real puzzles in the subject of partial differential equations lay in understanding what kinds of boundary or initial conditions different PDEs require. For example, it was known from the d'Alembert solution of the one-dimensional wave equation (Volume 1, sections 11.5.7–8) that both the function u and its normal derivative had to be specified on a line in the z–t plane in order to determine the solution u uniquely. On the other hand, it was also known that, to specify a solution of Laplace's equation uniquely, it was enough to give either the function value (for example, the electrostatic potential) *or* the normal derivative (for example, the electric field) on a boundary. It is clear on physical grounds that specifying both the potential and the field on a boundary overdetermines the problem, possibly leading to no solution.

Initial conditions such that the values of the function u and its normal derivative du/dn are given on a curve or surface Γ are called **Cauchy conditions**. Specifying the potential and the field on one boundary surface (such as the x axis for a two-dimensional electrostatics problem) leads to a unique solution, for example.

A theorem that was proved in general by Sofia Vasilyevna Kovalevskaya (1850–1891), a special case of which was proved earlier by Augustin Louis Cauchy (1789–1857), states that, for a partial differential equation that is analytic in the unknown function u and its partial derivatives, Cauchy initial conditions determine a unique solution.

Useful boundary conditions include **Dirichlet conditions**, in which only the value of u is prescribed on a closed curve or surface Γ, and **Neumann conditions**, in which only the value of du/dn is prescribed.

16.3.4 Characteristic equation

There are three equations that we can use to determine the three second partial derivatives of u given u and du/dn on Γ, the first partial derivatives that we just computed, and the partial differential equation itself:

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) &= \frac{dx}{ds}\frac{\partial^2 u}{\partial x^2} + \frac{dy}{ds}\frac{\partial^2 u}{\partial x \partial y} \\ \frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) &= \frac{dx}{ds}\frac{\partial^2 u}{\partial x \partial y} + \frac{dy}{ds}\frac{\partial^2 u}{\partial y^2} \\ -e &= a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} \end{aligned} \tag{16.74}$$

This system of three linear equations in the three unknowns $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$, and $\partial^2 u/\partial y^2$ has a unique solution if and only if the determinant of coefficients,

$$\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \\ a & b & c \end{vmatrix} = a\left(\frac{dy}{ds}\right)^2 - b\frac{dy}{ds}\frac{dx}{ds} + c\left(\frac{dx}{ds}\right)^2, \tag{16.75}$$

is non-zero. To make the situation when this condition fails easier to analyze, we take advantage of the fact that, along Γ,

$$\frac{dy}{dx} = \frac{dy}{ds}\Big/\frac{dx}{ds}. \tag{16.76}$$

After we multiply the right-hand side of (16.75) by $(dx/ds)^{-2}$, then, we obtain

$$\boxed{a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0} \tag{16.77}$$

as the condition for the vanishing of the determinant of the coefficients of the linear system (16.74). Eq. (16.77), the **characteristic equation** of the PDE (16.63), is the condition that there is *not* a unique solution for the second-order partial derivatives in terms of data on the curve Γ.

The roots of the characteristic equation (16.77) are

$$\frac{dy}{dx} = \lambda = \frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}. \tag{16.78}$$

Each distinct root defines a differential equation, the solution of which is a family of **characteristic curves** C in the x–y plane.

We have shown that it is not possible to compute the partial derivatives of second order given only data on a characteristic curve.

16.3.5 Classification

There are three different types of second-order quasilinear PDEs in two independent variables, depending on the nature of the roots given in Eq. (16.78):

$$\text{If } b^2 - 4ac \text{ is } \begin{cases} > 0, & \text{the PDE is } \textbf{hyperbolic}; \\ = 0, & \text{the PDE is } \textbf{parabolic}; \\ < 0, & \text{the PDE is } \textbf{elliptic}. \end{cases} \tag{16.79}$$

$b^2 - 4ac > 0$: The characteristic equation has two real roots, λ_1 and λ_2. There are two real, distinct families of characteristic curves with slopes equal to λ_1 and λ_2.

Example: The wave equation,

$$\frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{16.80}$$

in which $a = 1$, $b = 0$, $c = -1/v^2$, $b^2 - 4ac = 4/v^2$, $\lambda_1 = 1/v$, and $\lambda_2 = -1/v$. The equations of characteristic curves for the wave equation are $t = z/v + t_1$ and $t = -z/v + t_2$.

Hyperbolic equations require Cauchy initial conditions on a non-characteristic curve.

$b^2 - 4ac = 0$: The characteristic equation has one real root, $\lambda = b/2a$. There is one real family of characteristic curves.

Example: The diffusion equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{16.81}$$

in which $a = 1$, $b = 0$, $c = 0$, $b^2 - 4ac = 0$, and $\lambda = 0$. The equation of a characteristic curve in this example is $t = t_0$.

Parabolic equations can be solved using either Cauchy initial conditions on a non-characteristic curve, or Dirichlet or Neumann boundary conditions.

$b^2 - 4ac < 0$: The characteristic equation has two complex conjugate roots. There are no real characteristic curves.

Example: Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{16.82}$$

in which $a = 1$, $b = 0$, $c = 1$, and $b^2 - 4ac = -4$.

Since there are no real characteristic curves for an elliptic PDE, it is always possible to determine the second partial derivatives from Cauchy initial conditions on a curve, or Dirichlet or Neumann boundary conditions.

Exercises for Section 16.3

16.3.1 Show that

$$u(x,t) = e^{-t}(A \sin x + B \cos x), \tag{16.83}$$

where A and B are constants, is a solution of the diffusion Eq. (16.81). Discuss physically useful initial conditions on a non-characteristic curve, and conditions on a boundary that may include a segment of a characteristic curve, that determine the values of A and B uniquely.

16.3.2 Show that

$$u(x,t) = \int_{-\infty}^{\infty} c(k) e^{ikx - k^2 t} dk \tag{16.84}$$

is a solution of the diffusion Eq. (16.81). Discuss initial and boundary conditions that determine the function c uniquely.

16.3.3 Show that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x') \exp\left[-\frac{(x-x')^2}{4t}\right] dx' \tag{16.85}$$

is a solution of the diffusion Eq. (16.81) such that

$$u(x,0) = f(x). \tag{16.86}$$

What boundary or initial conditions are used in this solution?

16.4 Hyperbolic PDEs

16.4.1 Characteristic differential equations

The goal here is to express the original hyperbolic PDE as two differential equations for the characteristic curves, plus a differential equation that describes how u changes along a curve.

From the first two equations of (16.74) one can solve for $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ in terms of $\partial^2 u/\partial x \partial y$, obtaining

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial x}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dy}{ds}}{\dfrac{dx}{ds}} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dx}{ds}}{\dfrac{dy}{ds}}. \end{aligned} \tag{16.87}$$

In order to eliminate $\partial^2 u/\partial x \partial y$, we substitute these expressions into the PDE:

$$a\frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial x}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dy}{ds}}{\dfrac{dx}{ds}} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dx}{ds}}{\dfrac{dy}{ds}} = -e. \tag{16.88}$$

Clearing by multiplying through by $(dx/ds)(dy/ds)$ and then collecting terms, one gets

$$\begin{aligned} &\frac{\partial^2 u}{\partial x \partial y}\left[-a\left(\frac{dy}{ds}\right)^2 + b\frac{dx}{ds}\frac{dy}{ds} - c\left(\frac{dx}{ds}\right)^2\right] \\ &\qquad = \left[a\frac{dy}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) + c\frac{dx}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) + e\frac{dx}{ds}\frac{dy}{ds}\right]. \end{aligned} \tag{16.89}$$

The first line of Eq. (16.89) vanishes on a characteristic curve C. Therefore, on C, both of the equations

$$\begin{aligned} a\frac{dy}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) + c\frac{dx}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) + e\frac{dx}{ds}\frac{dy}{ds} = 0 \\ \text{and} \quad \frac{dy}{ds} = \lambda\frac{dx}{ds}, \end{aligned} \tag{16.90}$$

must be obeyed, where λ is one of the roots of the characteristic equation (16.77) given in Eq. (16.78). Substituting for dy/ds and cancelling dx/ds, one obtains the **characteristic differential equations** (for $i = 1, 2$)

$$\boxed{\lambda_i \frac{a}{c}\left(\frac{d}{ds}\right)_i \left(\frac{\partial u}{\partial x}\right) + \left(\frac{d}{ds}\right)_i \left(\frac{\partial u}{\partial y}\right) = -\lambda_i \frac{e}{c}\left(\frac{dx}{ds}\right)_i.} \tag{16.91}$$

Eqs. (16.91) and the ordinary differential equation

$$\boxed{\frac{du}{ds} = \frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dy}{ds}\frac{\partial u}{\partial y}} \tag{16.92}$$

(which describes the variation of u along any curve, including a characteristic curve) determine u, given Cauchy initial conditions.

16.4.2 Characteristic coordinates for the one-dimensional wave equation

The **one-dimensional wave equation** is

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) u(z,\, t) = 0. \tag{16.93}$$

For this equation, $a = 1$, $b = 0$, $c = -1/v^2$, and $e = 0$, making the eigenvalue equation (16.78) read

$$\frac{dt}{dz} = \lambda = \pm\frac{1}{2}\sqrt{4/v^2}. \tag{16.94}$$

It follows that

$$\lambda_1 = \frac{1}{v}, \quad \lambda_2 = -\frac{1}{v} \tag{16.95}$$

and therefore the equations of the characteristic curves are

$$z \pm vt = k \tag{16.96}$$

where k is any real number.

Let

$$\xi = z + vt, \quad \eta = z - vt. \tag{16.97}$$

The equations $\xi = k$ and $\eta = k'$ are the equations of characteristic curves. As is true for hyperbolic PDEs in general, there are two families of characteristic curves:

- Family 1, consisting of the lines $\eta = z - vt =$ constant
- Family 2, consisting of the lines $\xi = z + vt =$ constant

The element of arc length measured along a characteristic curve $z + vt = k$ or $z - vt = k'$ obeys the equation $ds^2 = dz^2 + (vdt)^2$. Therefore

$$ds = \pm\, dz \left[1 + v^2\left(\frac{dt}{dz}\right)^2\right]^{\frac{1}{2}} = \pm\sqrt{2}\, dz. \tag{16.98}$$

For family 1,

$$\lambda = \lambda_1 = \frac{1}{v}, \quad (ds)_1 = \sqrt{2}\, dz = \frac{1}{\sqrt{2}}\, d\xi\,; \tag{16.99}$$

for family 2,

$$\lambda = \lambda_2 = -\frac{1}{v}, \quad (ds)_2 = -\sqrt{2}\, dz = -\frac{1}{\sqrt{2}}\, d\eta, . \tag{16.100}$$

The **canonical variables** ξ, η, or, equivalently, the **characteristic variables**

$$\begin{aligned} s_1 &= \frac{1}{\sqrt{2}}\, \xi\,, \\ s_2 &= -\frac{1}{\sqrt{2}}\, \eta\,, \end{aligned} \tag{16.101}$$

define a coordinate net for the solution of the wave equation.

16.4.3 Solution of the homogeneous one-dimensional wave equation by the method of characteristics

Recalling that $a = 1$, $b = 0$, $c = -1/v^2$, and $e = 0$, one sees that for the one-dimensional wave equation the characteristic differential equations (16.91) take on the forms

$$\left(\frac{d}{ds}\right)_1 \left(\frac{\partial u}{\partial z} - \frac{1}{v}\frac{\partial u}{\partial t}\right) = 0, \quad \left(\frac{d}{ds}\right)_2 \left(\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t}\right) = 0. \tag{16.102}$$

To express these equations in terms of canonical or characteristic variables, it is convenient to make use of the chain rule for partial differentiation, which implies that

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{1}{v}\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}. \tag{16.103}$$

Then

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right), \quad \frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\right). \tag{16.104}$$

Let

$$w(\xi,\, \eta) := u(z,\, t). \tag{16.105}$$

With Eqs. (16.104) and (16.105), the characteristic differential equations become

$$\frac{d}{d\xi}\left(\frac{\partial w}{\partial \eta}\right) = 0, \quad \frac{d}{d\eta}\left(\frac{\partial w}{\partial \xi}\right) = 0. \tag{16.106}$$

One can see at once that the general solution of Eqs. (16.106) is

$$\frac{\partial w}{\partial \eta} = f'(\eta), \quad \frac{\partial w}{\partial \xi} = g'(\xi) \tag{16.107}$$

where the functions f and g belong to $\mathcal{C}^2\big((-\infty,\, \infty);\, \mathbb{R}\big)$ but are otherwise arbitrary. Therefore the general solution of the one-dimensional wave equation (16.93) is

$$w(\xi, \eta) = f(\eta) + g(\xi) \Rightarrow u(z,\, t) = f(z - vt) + g(z + vt). \tag{16.108}$$

The wave represented by f travels in the $+z$ direction, since f takes on the same value, $f(\xi_0)$, for every spacetime point $(z,\, t)$ such that $z - vt = \xi_0$. Similarly, the wave represented by g travels in the $-z$ direction.

Because the characteristic curves for Eq. (16.93) are independent of the wave amplitude u, the ordinary differential equation for

16.4.4 Cauchy problem for a string of infinite length

For a string of infinite length two initial conditions are necessary in order to determine the two independent functions f and g in Eq. (16.108). Let

$$d(z) := u(z,\, 0), \quad s(z) := \frac{1}{v}\frac{\partial u}{\partial t}(z,\, t)\bigg|_{t=0}. \tag{16.109}$$

The function d specifies the initial displacement profile of the wave, while the function s specifies the x-component of the velocity of the string itself, expressed as a fraction of the phase velocity of waves on the string. It follows that

$$d(z) = f(z) + g(z) \tag{16.110}$$

and that

$$s(z) = -f'(z) + g'(z) \Rightarrow -f(z) + g(z) = \int_0^z s(z')\, dz'. \tag{16.111}$$

From Eq. (16.108) and Eqs. (16.109–16.111) one finds that the general solution of the wave equation for an infinitely long string is

$$u(z,\, t) = \tfrac{1}{2}\left[d(z - vt) + d(z + vt) + \int_{z-vt}^{z+vt} s(z')\, dz'\right]. \tag{16.112}$$

Fig. 16.6 illustrates the fact that the value of u at the spacetime point $(z,\, t)$ depends upon the values of d at the points $(z \pm vt,\, 0)$ and the values of $\partial u/\partial t$ on the interval $[z - vt,\, z + vt]$ along the z-axis. The straight lines with slopes ± 1 which intersect at $(z,\, t)$ are examples of *characteristic curves* for the wave equation Eq. (16.93). The triangular spacetime region defined by the points $(z,\, t)$ and $(z \pm vt,\, 0)$ is called the **domain of dependence**, because we have shown that the value of u at $(z,\, t)$ depends upon $\partial u/\partial t$ along any curve which traverses this domain and intersects both characteristic curves.

Volume 1, section 11.5.9 discusses the solution of Eq. (16.93) for a string of finite length.

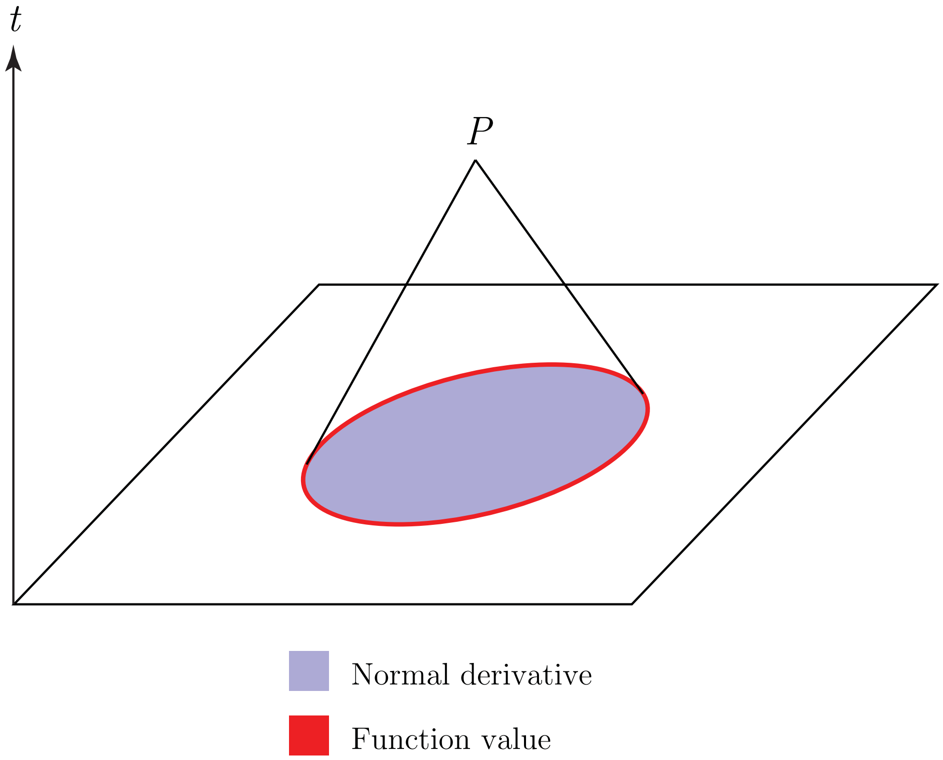

Figure 16.6: Cauchy initial conditions for the wave equation. The value of the unknown function u at the point P is determined by the values of u on the red curve and the values of the normal derivative of u on the blue region.

Exercises for Section 16.4

16.4.1 (To be done)

16.5 Parabolic PDEs

Exercises for Section 16.5

16.5.1 (To be done)