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# DIGITAL FILTERS

Notes prepared for EE 6481

by

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# DIGITAL FILTERS (1)

- Consider an input data stream with values $u_n = u(t_n)$ and an output data stream with values $y_n$
  - ▷ If
$$y_n = \sum_{k=-N}^{N} c_k \, u_{n-k}$$
    then $c_{-N}, \ldots, c_N$ define a **non-recursive digital filter** with constant coefficients
    - ○ $y_n$ is the discrete convolution of the coefficients $c_k$ with the input data
    - ○ Response to a unit impulse at $t = t_n$ is the output stream $(\ldots, 0, \; c_{-N}, \ldots, c_N, 0, \ldots, )$
    - ○ Example: Straight-line differentiating filter
$$y_n = -\frac{\sum_{k=-N}^{N} k u_{n-k}}{\sum_{m=-N}^{N} m^2}$$

# DIGITAL FILTERS (2)

- Computation for $n = 2$:

$$
y_n = \begin{array}{cccc} & \vdots & & \\ + & u_{n-3} & \cdot & 0 \\ + & u_{n-2} & \cdot & c_2 \\ + & u_{n-1} & \cdot & c_1 \\ + & u_n & \cdot & c_0 \\ + & u_{n+1} & \cdot & c_{-1} \\ + & u_{n+2} & \cdot & c_{-2} \\ + & u_{n+3} & \cdot & 0 \\ & \vdots & & \end{array}
$$

▷ Visualize the computation of $y_m$ as sliding the coefficients $c_N, \ldots, c_{-N}$ so that $c_0$ is next to $u_m$, then multiplying and adding

# DIGITAL FILTERS (3)

- Every non-recursive digital filter with constant coefficients can be expressed as the sum of a symmetric and an antisymmetric filter:

$$c_k = a_k + s_k$$

where

$$s_k = \tfrac{1}{2}\big(c_k + c_{-k}\big), \quad a_k = \tfrac{1}{2}\big(c_k - c_{-k}\big),$$

▷ Symmetric: $s_{-k} = s_k$

- Smoothing (integrating) filters
- Example: Centered moving average

$$y_n = \frac{1}{2N+1} \sum_{k=-N}^{N} u_{n-k},$$

▷ Antisymmetric: $a_{-k} = a_k$

- Differentiating filters

# DIGITAL FILTERS (4)

- Noise amplification by a non-recursive digital filter
  - ▷ Suppose that we can only measure $u_k + e_k$, where $e_k$ is an error that is uncorrelated with $e_l$ if $l \neq k$:

$$\forall\, k \in \mathbb{Z} : \begin{cases} E[e_k] = 0, \\ E[e_k e_l] = \sigma^2 \delta_k^l \end{cases}$$

  - ▷ Expected value of $y_m$ is independent of error in input:

$$E[y_n] = E\left[\sum_{-N}^{N} c_k(u_{n-k} + e_{n-k})\right] = \sum_{-N}^{N} c_k u_{n-k}$$

# DIGITAL FILTERS (5)

- Noise amplification by a non-recursive digital filter
  - ▷ Variance of $y_m$:

$$E\left[\left(y_n - E[y_n]\right)^2\right] = E\left[\left(\sum_{-N}^{N} c_k e_{n-k}\right)^2\right]$$

$$= E\left[\sum_{k=-N}^{N} \sum_{m=-N}^{N} c_k c_m e_{n-k} e_{n-m}\right]$$

$$= \sigma^2 \sum_{k=-N}^{N} c_k^2$$

  - ▷ $\sum_{k=-N}^{N} c_k^2$ is the **noise amplification factor** of the digital filter
    - ∘ One would like to have $\sum_{k=-N}^{N} c_k^2 \leq 1$
    - ∘ For the centered moving average, the factor is $(2N+1)^{-1}$

# DIGITAL FILTERS (6)

- Transfer function of a non-recursive digital filter with constant coefficients
  - ▷ Because the filter represents a linear, time-shift invariant system, an input sinusoid results in an output sinusoid at the same frequency
  - ▷ Let the input be

$$u_m^{(\omega)} = u^{(\omega)}(t_m) = e^{jm\omega h}$$

where $t_m = mh$

  - ▷ The output is

$$y_m^{(\omega)} = \sum_{k=-N}^{N} c_k e^{j(m-k)\omega h} = H(\omega h)\, u_m^{(\omega)}$$

where the **transfer function** is

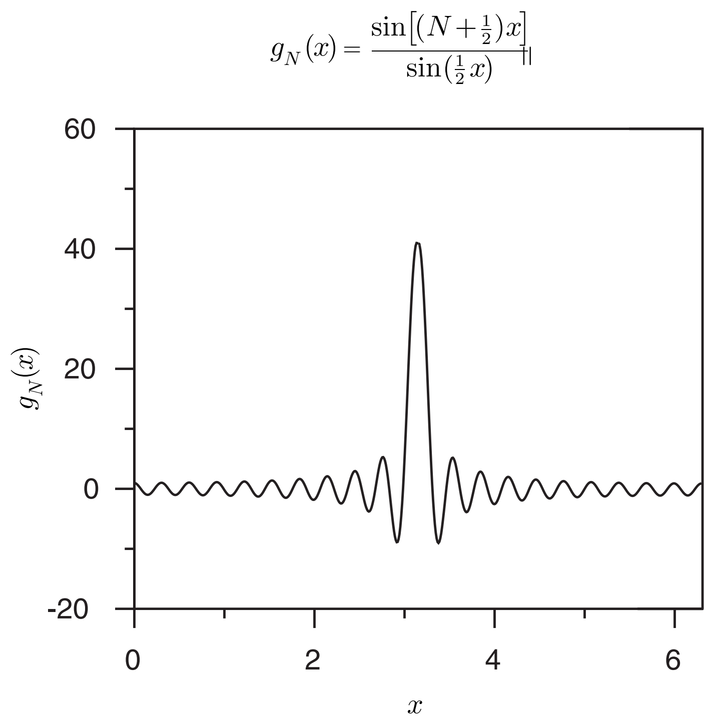
$$\boxed{H(\omega h) = \frac{y_m^{(\omega)}}{u_m^{(\omega)}} = \sum_{k=-N}^{N} c_k e^{-jk\omega h}}$$

# DIGITAL FILTERS (7)

- Transfer function of a centered moving average

$$H(\omega h) = \frac{1}{2N+1} \sum_{k=-N}^{N} e^{-jk\omega h} = \frac{1}{2N+1} g_N(\omega h)$$

  ▷ Plot of the grating function $g_N(\omega h)$ for $N = 20$:



# DIGITAL FILTERS (8)

- Consider an input data stream with values $u_n = u(t_n)$ and an output data stream with values $y_n$
  - ▷ If
$$y_n = \sum_{k=-N}^{N} c_k u_{n-k} + \sum_{k=1}^{M} d_k y_{n-k}$$
    then $c_{-N}, \ldots, c_N$ and $d_1, \ldots, d_M$ define a **causal recursive digital filter** with constant coefficients
    - ∘ Example: Trapezoidal rule for numerical integration
$$y_n = y_{n-1} + \tfrac{1}{2}h\big(u_n + u_{n-1}\big)$$
      - ◇ $u_n$ is the sampled value of the integrand at $t_n$
      - ◇ $y_n$ is the value of the computed integral up to $t_n$
      - ◇ Nonzero filter coefficients are
$$c_0 = \frac{h}{2}, \quad c_1 = \frac{h}{2}, \quad d_1 = 1$$

# DIGITAL FILTERS (9)

- Transfer function of a causal, recursive digital filter with constant coefficients
  - ▷ Because the filter represents a linear, time-shift invariant system, an input sinusoid results in an output sinusoid at the same frequency
  - ▷ Let the input be

$$u_m^{(\omega)} = u^{(\omega)}(t_m) = e^{jm\omega h}$$

    where the sampling times are $t_m = mh$
    - ∘ The output is also single-frequency: $y_{m-k}^{(\omega)} = y_m^{(\omega)} e^{-jk\omega h}$
  - ▷ The output is

$$\begin{aligned} y_m^{(\omega)} &= \sum_{k=-N}^{N} c_k\, e^{j(m-k)\omega h} + \sum_{k=1}^{M} d_k y_{m-k}^{(\omega)} \\ &= u_m^{(\omega)} \sum_{k=-N}^{N} c_k\, e^{-jk\omega h} + y_m^{(\omega)} \sum_{k=1}^{M} d_k\, e^{-jk\omega h} \end{aligned}$$

# DIGITAL FILTERS (10)

- Transfer function of a causal, recursive digital filter with constant coefficients
  - ▷ Collect terms in $y_m^{(\omega)}$ in last equation on preceding slide
  - ▷ The transfer function is

$$H(\omega h) = \frac{y_m^{(\omega)}}{u_m^{(\omega)}} = \frac{\displaystyle\sum_{k=-N}^{N} c_k \, e^{-jk\omega h}}{1 - \displaystyle\sum_{k=1}^{M} d_k \, e^{-jk\omega h}}$$

  - ▷ Example: Trapezoidal rule ($c_0 = \frac{h}{2}, \ c_1 = \frac{h}{2}, \ d_1 = 1$)

$$H(\omega h) = \left(\frac{h}{2}\right) \frac{1 + e^{-j\omega h}}{1 - e^{-j\omega h}} = \left(\frac{h}{2j}\right) \frac{\cos(\omega h/2)}{\sin(\omega h/2)}$$

# DIGITAL FILTERS (11)

- Transfer function of an integrating digital filter with constant coefficients
  - ▷ The transfer function of an ideal integrator is $1/j\omega$ :

$$\text{input} = e^{j\omega t}, \text{ output} = \int e^{j\omega t} dt = \frac{1}{j\omega} e^{j\omega t} \Rightarrow \frac{\text{output}}{\text{input}} = \frac{1}{j\omega}$$

  - ▷ The **transfer-function ratio** (TFR) is

$$j\omega \times \text{transfer function of integrating filter}$$

    - ○ Ratio of actual to ideal transfer functions
  - ▷ Example 1: Trapezoidal rule ($c_0 = \frac{h}{2},\ c_1 = \frac{h}{2},\ d_1 = 1$)

$$\text{TFR}_{\text{trap}} = j\omega H(\omega h) = \left(\frac{\omega h}{2}\right) \frac{\cos(\omega h/2)}{\sin(\omega h/2)}$$

# DIGITAL FILTERS (12)

- Transfer function of an integrating digital filter with constant coefficients
  - ▷ Example 2: Midpoint rule ($c_1 = 2h,\ d_2 = 1$)

$$H(\omega h) = 2h\,\frac{e^{-j\omega h}}{1 - e^{-2j\omega h}} = 2h\,\frac{1}{2j\sin(\omega h)} = \left(\frac{h}{j}\right)\frac{1}{\sin(\omega h)}$$

$$\mathrm{TFR_{mid}} = j\omega H(\omega h) = \frac{\omega h}{\sin(\omega h)}$$

  - ▷ Example 3: Left-hand rectangle rule ($c_1 = h,\ d_1 = 1$)

$$H(\omega h) = h\,\frac{e^{-j\omega h}}{1 - e^{-j\omega h}} = h\,\frac{e^{-j\omega h/2}}{2j\sin(\omega h/2)} = \left(\frac{h}{2j}\right)\frac{e^{-j\omega h/2}}{\sin(\omega h/2)}$$

$$\mathrm{TFR_{left}} = j\omega H(\omega h) = \left(\frac{\omega h}{2}\right)\frac{e^{-j\omega h/2}}{\sin(\omega h/2)}$$

# DIGITAL FILTERS (13)

- Transfer function of an integrating digital filter with constant coefficients
  - ▷ Example 3: Simpson's rule

$$y_n = y_{n-1} + \tfrac{1}{3}h\big(u_n + 4u_{n-1} + u_{n-2}\big)$$

  - ▷ Coefficients: $c_0 = h/3,\ c_1 = 4h/3,\ c_2 = h/3,\ d_2 = 1$
  - ▷ Transfer function:

$$H(\omega h) = \left(\frac{h}{j3}\right)\frac{2+\cos(\omega h)}{\sin(\omega h)}$$

  - ▷ Transfer function ratio:

$$\mathrm{TFR}_{\mathrm{Sim}} = j\omega H(\omega h) = \left(\frac{\omega h}{3}\right)\frac{2+\cos(\omega h)}{\sin(\omega h)}$$

## Transfer-function ratio $|j\omega H(\omega h)|$ for selected integration rules