# PARAXIAL WAVE EQUATION IN BULK MEDIA

- The **paraxial wave equation**

$$\left[\nabla_T^2 + 2ik\frac{\partial}{\partial z'}\right]\mathcal{E} = \frac{4\pi i k^2}{n^2}\mathcal{P}$$

is an approximate form of the wave equation that applies when:

  ▷ The electric field can be expressed as the product of a rapidly varying "carrier"

$$e^{-i\omega t' + i\phi} = e^{i(kz - \omega t + \phi)}$$

  (where $t' = t - nz/c$) and a slowly varying phasor "signal" $\mathcal{E}$:

$$\mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{e}}|\mathcal{E}|e^{-i\omega t' + i\phi}\right]$$

  ▷ It is known on physical grounds that only propagation vectors near a fixed direction ($z$) are signficant

- Applications: Laser-beam and laser-pulse propagation

# PARAXIAL WAVE EQUATION IN BULK MEDIA (1)

- Eliminate $\mathbf{H} = \mathbf{B}$ from the Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$$

by calculating $\nabla \times (\nabla \times \mathbf{E})$ and making the approximation

$$\nabla(\nabla \cdot \mathbf{E}) = -4\pi\nabla(\nabla \cdot \mathbf{P}) = \mathbf{0}$$

(Exact if $\epsilon$ is a constant; OK for weak nonlinearities)

- Wave equation:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = \frac{4\pi\sigma}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

# PARAXIAL WAVE EQUATION IN BULK MEDIA (2)

- Introduce slowly varying envelope functions:

$$\mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{e}}\mathcal{E}(\mathbf{r}_T,\, z,\, t)e^{i(kz-\omega t)}\right] = \mathrm{Re}\left[\hat{\mathbf{e}}\mathcal{E}e^{-i\omega t'}\right]$$

$$\begin{aligned}\mathbf{P}_{NL} &= \mathrm{Re}\left[-i\hat{\mathbf{e}}\mathcal{P}(\mathbf{r}_T,\, z,\, t)e^{i(kz-\omega t)}\right] \\ &= \mathrm{Re}\left[\hat{\mathbf{e}}(C-iS)e^{-i\omega t'+i\phi}\right]\end{aligned}$$

where $k = n\omega/c$,

$$t' := t - \frac{nz}{c} = \text{retarded time}, \quad \text{and} \quad \mathcal{E} = |\mathcal{E}|e^{i\phi}$$

It follows that

$$\boxed{\mathcal{P} = i(C-iS)e^{i\phi}}$$

where $C$ produces dispersion, and $S$ produces absorption or gain

# PARAXIAL WAVE EQUATION IN BULK MEDIA (3)

- **Slowly varying envelope approximation**: Assume that the envelopes $\mathcal{E}$ and $\mathcal{P}$ vary slowly on the scale of an optical wavelength or an optical period:

$$\left|\frac{\partial \mathcal{E}}{\partial z}\right| \ll k|\mathcal{E}|; \quad \left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll \omega|\mathcal{E}|$$

  - ▷ In the wave equation:
    - ∘ Neglect $\partial^2\mathcal{E}/\partial z^2$ compared to $k\ \partial\mathcal{E}/\partial z$
    - ∘ Neglect $\partial^2\mathcal{E}/\partial t^2$ compared to $\omega\ \partial\mathcal{E}/\partial t$
    - ∘ Keep $k\ \partial\mathcal{E}/\partial z$ and $(n\omega/c)\partial\mathcal{E}/\partial t$ because $[k^2 - (n\omega/c)^2]\mathcal{E} = 0$
  - ▷ Resulting "one-way" wave equation:

$$\left[\nabla_T^2 + 2ik\left(\frac{\partial}{\partial z} + \frac{n}{c}\frac{\partial}{\partial t}\right)\right]\mathcal{E} = \frac{4\pi i k^2}{n^2}\mathcal{P}$$

# PARAXIAL WAVE EQUATION IN BULK MEDIA (4)

- Transform to a frame moving with the pulse:

$$t' := t - \frac{nz}{c}$$
$$z' := z$$

  - ▷ Functions transform as follows:

$$\bar{f}(\mathbf{r}_T,\, z',\, t') := f(\mathbf{r}_T,\, z,\, t)$$
$$\frac{\partial \bar{f}}{\partial z'} = \frac{\partial f}{\partial z} + \frac{n}{c}\frac{\partial f}{\partial t}$$

- Resulting **paraxial** or **one-way wave equation**:

$$\boxed{\left[\nabla_T^2 + 2ik\frac{\partial}{\partial z'}\right]\overline{\mathcal{E}} = \frac{4\pi i k^2}{n^2}\overline{\mathcal{P}}}$$

where $\overline{\mathcal{E}}$ and $\overline{\mathcal{P}}$ are functions of $\mathbf{r}_T$, $z'$ and $t'$

# PARAXIAL WAVE EQUATION IN BULK MEDIA (5)

- Complex polarization amplitude:

$$\mathcal{P} = i(C - iS)e^{i\phi}$$

  ▷ Real polarization (dipole moment per unit volume):

$$\mathbf{P} = \mathrm{Re}\left[\hat{\mathbf{e}}(C - iS)e^{-i\omega t' + i\phi}\right]$$

  ▷ If $\mathcal{P}$ is purely dispersive ($S = 0$; no absorption or gain) then the real polarization and electric field are

$$\mathbf{P} = \mathrm{Re}\left[\hat{\mathbf{e}}Ce^{-i\omega t' + i\phi}\right], \quad \mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{e}}|\mathcal{E}|e^{-i\omega t' + i\phi}\right]$$

- **Electric susceptibility** $\chi$:

$$\mathbf{P} = \chi\mathbf{E} \quad \Rightarrow \quad \boxed{\chi = \frac{C}{|\mathcal{E}|} = \frac{\mathcal{P}}{i\mathcal{E}}}$$

# PARAXIAL WAVE EQUATION IN BULK MEDIA (6)

- Interpretation of the "disappearance" of $\partial/\partial t'$ from the paraxial wave equation

$$\left[\nabla_T^2 + 2ik\frac{\partial}{\partial z'}\right]\mathcal{E} = \frac{4\pi i k^2}{n^2}\mathcal{P}$$

  - ▷ Both the polarization $\mathcal{P}$ and the field $\mathcal{E}$ still depend on the retarded time. A steady-state approximation has **not** been made in going to a comoving frame.
  - ▷ A steady-state or other dynamical approximation may be applied in the calculation of $\mathcal{P}$.
  - ▷ If $\mathcal{P}$ is calculated by a method that allows for coherent or dispersive effects, then different retarded times are coupled through $\mathcal{P}$.

# PARAXIAL WAVE EQUATION IN BULK MEDIA (7)

- Equivalence of the homogeneous paraxial wave equation (HPWE)

$$\frac{\partial \mathcal{E}}{\partial z} = \frac{i}{2k} \nabla_T^2 \mathcal{E}$$

to the Fresnel limit of the Kirchhoff diffraction integral:

▷ The formal solution of the HPWE is

$$\mathcal{E}(\mathbf{r}_T,\, z) = \exp\left\{ \frac{iz}{2k} \nabla_T^2 \right\} \mathcal{E}(\mathbf{r}_T,\, 0)$$

▷ In terms of the Fourier transform

$$\mathcal{E}(\mathbf{r}_T,\, 0) = (2\pi)^{-1} \int e^{i\mathbf{q}\cdot\mathbf{r}_T}\, \tilde{\mathcal{E}}(\mathbf{q},\, 0)\, d^2q,$$

the formal solution becomes

$$\mathcal{E}(\mathbf{r}_T,\, z) = (2\pi)^{-1} \int \exp\left\{ i \left[ -\frac{q^2 z}{2k} + \mathbf{q}\cdot\mathbf{r}_T \right] \right\} \tilde{\mathcal{E}}(\mathbf{q},\, 0)\, d^2q$$

$$= \text{Fourier transform of a product}$$

# PARAXIAL WAVE EQUATION IN BULK MEDIA (8)

- Convolution theorem:

$$(2\pi)^{-1}\int e^{i\mathbf{q}\cdot\mathbf{r}_T}\,\tilde{f}(\mathbf{q})\tilde{g}(\mathbf{q})\,d^2q = (2\pi)^{-1}\int f(\mathbf{r}_T-\mathbf{r}'_T)g(\mathbf{r}'_T)\,d^2r'_T$$

where

$$\tilde{f}(\mathbf{q}) := (2\pi)^{-1}\int e^{-i\mathbf{q}\cdot\mathbf{r}_T}\,f(\mathbf{r}_T)\,d^2r_T$$

- The formal solution of the HPWE, the Fourier transform

$$(2\pi)^{-1}\int \exp\left\{i\left[-\frac{\mathbf{q}^2 z}{2k}+\mathbf{q}\cdot\mathbf{r}_T\right]\right\}\,d^2q = -i\frac{k}{z}\exp\left[i\frac{k\mathbf{r}_T^2}{2z}\right]$$

and the convolution theorem imply that

$$\boxed{\mathcal{E}(\mathbf{r}_T,\,z) = -i\frac{k}{2\pi z}\int \exp\left\{i\frac{k}{2z}(\mathbf{r}_T-\mathbf{r}'_T)^2\right\}\mathcal{E}(\mathbf{r}'_T,\,0)\,d^2r'_T}$$

# PARAXIAL WAVE EQUATION IN BULK MEDIA (9)

- The Fresnel-Kirchhoff diffraction integral, evaluated in the forward direction, is

$$\mathcal{E}(\mathbf{r}_T,\, z)e^{ikz} = -i\frac{k}{2\pi}\int \frac{e^{iks}}{s}\mathcal{E}(\mathbf{r}'_T,\, 0)\, d^2r'_T$$

where

$$s^2 = (\mathbf{r}_T - \mathbf{r}'_T)^2 + z^2$$

- In the Fresnel limit

$$s \approx z + \frac{(\mathbf{r}_T - \mathbf{r}'_T)^2}{2z}$$

the Fresnel-Kirchhoff diffraction integral reduces to the integral form of the solution of the HPWE

# FREE-SPACE SOLUTION OF THE PARAXIAL WAVE EQUATION

- Lowest-order mode ($\mathrm{TEM}_{00}$):

$$\mathcal{E}(r, z', t') = \mathcal{E}_0(t') \cos \psi(z') e^{\Phi(r,z')}$$

where

$$\psi(z') = \tan^{-1} \frac{2(z' - z'_0)}{kw_0^2} = \tan^{-1} \frac{z' - z'_0}{l_R}$$

$$l_R = \frac{kw_0^2}{2} = \text{Rayleigh range}$$

$$\Phi(r, z') = -\left[\frac{r}{w(z')}\right]^2 [1 - i \tan \psi(z')] - i\psi(z')$$

$$w(z') = w_0 \sec \psi(z')$$