



ALIASING

Notes prepared for EE 6481

by

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ALIASING (1)

- Let f be a periodic, continuous-time function with period L :

$$\forall t \in \mathbb{R} : f(t + L) = f(t)$$

- ▷ Can also use this formula for **periodic extension** of a function that is defined only on a finite interval $[-L/2, L/2)$
- ▷ Expand f in a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t} \quad \text{where} \quad \omega_n = \frac{2\pi n}{L}$$

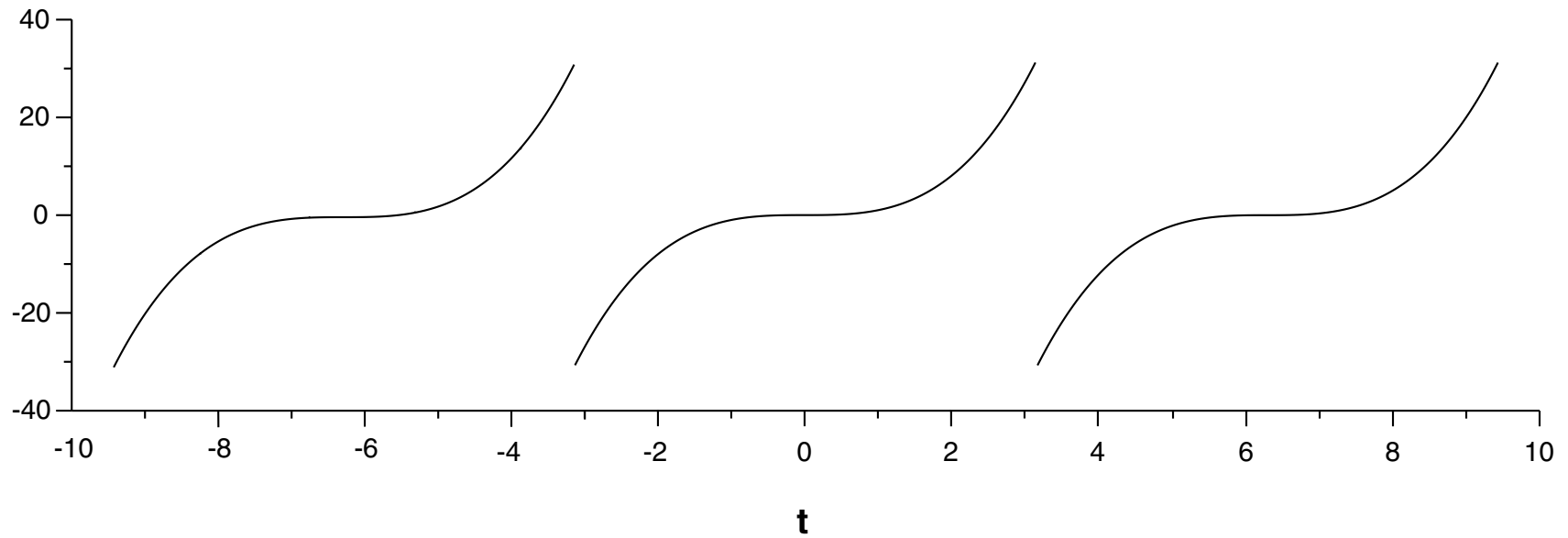
- One is using the **Fourier basis**

$$e^{j\omega_n t}, \quad n \in \mathbb{Z}$$

- Only the discrete frequencies ω_n enter, because f is periodic (or is defined on $[-L/2, L/2)$ and has been periodically extended)
- Calculation of the Fourier coefficients uses orthogonality:

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} e^{-j\omega_n t} f(t) dt$$

A PERIODICALLY EXTENDED FUNCTION



SAMPLING THEOREM (1)

- A function f is called **band-limited** with **bandwidth** B if the spectrum of f vanishes outside an interval of length $2B$ in transform (frequency) space
 - ▷ For a baseband signal, the interval in transform space is $[-B, B]$
- The sampling theorem asserts that a band-limited function can be reconstructed perfectly from samples taken at time intervals of $1/2B$
 - ▷ For a band-limited baseband signal,

$$f(t) = \int_{-B}^B \tilde{f}(\nu) e^{j2\pi\nu t} d\nu \quad \text{where} \quad \tilde{f}(\nu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\nu t} dt$$

- ▷ The Fourier transform \tilde{f} can be periodically extended to a function \tilde{f}_p with period $2B$

SAMPLING THEOREM (2)

- The periodic extension \tilde{f}_p has a Fourier series:

$$\tilde{f}_p(\nu) = \frac{1}{2B} \sum_m f_m e^{-j2\pi\nu t_m} \quad \text{where} \quad t_m = \frac{m}{2B}$$

$$\text{and} \quad f_m = \int_{-B}^B \tilde{f}_p(\nu) e^{j2\pi\nu t_m} d\nu = f(t_m)$$

▷ Then

$$\begin{aligned} f(t) &= \int_{-B}^B \tilde{f}_p(\nu) e^{j2\pi\nu t} d\nu = \frac{1}{2B} \sum_m f_m \int_{-B}^B e^{j2\pi\nu(t-t_m)} d\nu \\ &= \sum_m f_m \operatorname{sinc}(2B(t-t_m)) \quad \text{where} \quad \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \end{aligned}$$

- ▷ It follows that $f(t_n) = f_n$, so the function f is reconstructed at the sampling times $t_n = n/2B$
- ▷ The interval between successive sampling times is $1/2B$

ALIASING (2)

- Sample a continuous-time function f at the $2N$ discrete points

$$t = t_l = lh \quad \text{where} \quad h = \frac{L}{2N}$$

▷ This produces a **discrete-time function**, which is defined entirely by its values $f(t_{-N}), \dots, f(t_{N-1})$, where $t \in [-L/2, L/2)$

- Only a finite frequency range is meaningful for a discrete-time function, because

$$\omega_{n+2N} t_l = \left(\frac{2\pi(n+2N)}{2Nh} \right) lh = \omega_n t_l + 2\pi l$$

- Therefore ω_n is confined to the **Nyquist interval**,

$$-\frac{1}{2}\omega_N \leq \omega_n < \frac{1}{2}\omega_N,$$

where

$$\omega_N = \frac{2\pi}{h} = \text{Nyquist frequency}$$

ALIASING (3)

- A discrete-time function f is a vector with $2N$ components
 - ▷ Only $2N$ basis vectors are needed to expand a $2N$ -component vector
 - ▷ Possible basis sets:
 - Canonical basis (one component equals 1, all others equal 0)
 - Sampled Fourier basis $u_n(t_l) = e^{j\omega_n t_l}$
- Orthogonality of the sampled Fourier basis functions $u_n(t_l)$:

$$\sum_{l=-N}^{N-1} e^{j(\omega_n - \omega_m)t_l} = \begin{cases} 2N, & \text{if } n = m + 2Nq; \\ 0, & \text{otherwise} \end{cases}$$

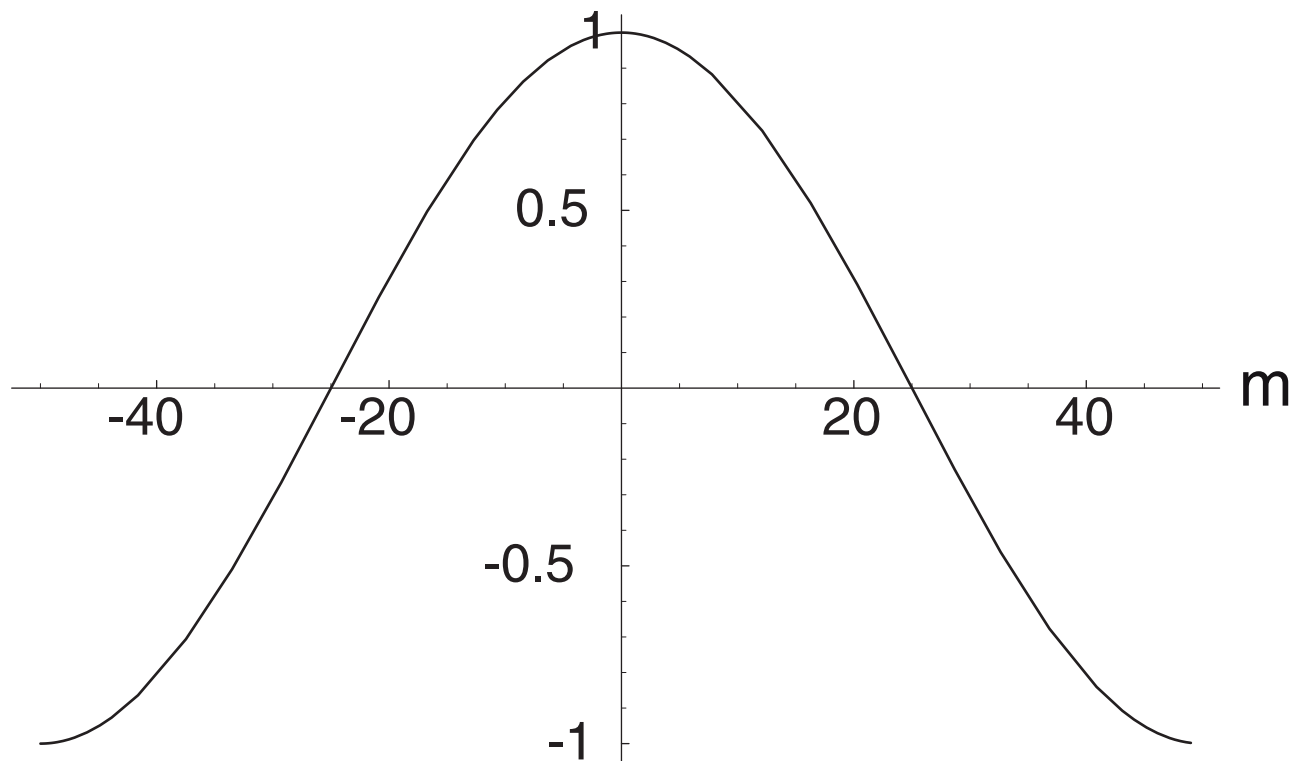
- Expansion in the sampled Fourier basis \Rightarrow **discrete Fourier expansion:**

$$f(t_l) = \sum_{n=-N}^{N-1} C_n e^{j\omega_n t_l} \quad \text{where} \quad C_n = \frac{1}{2N} \sum_{l=-N}^{N-1} f(t_l) e^{-j\omega_n t_l}$$

SAMPLED FOURIER BASIS COMPONENTS (1)

- Values of $\text{Re}[u_n(t_m)] = e^{j\omega_n t_m}$ for $N = 50$, $n = 1$ and $m \in [-N, N - 1]$

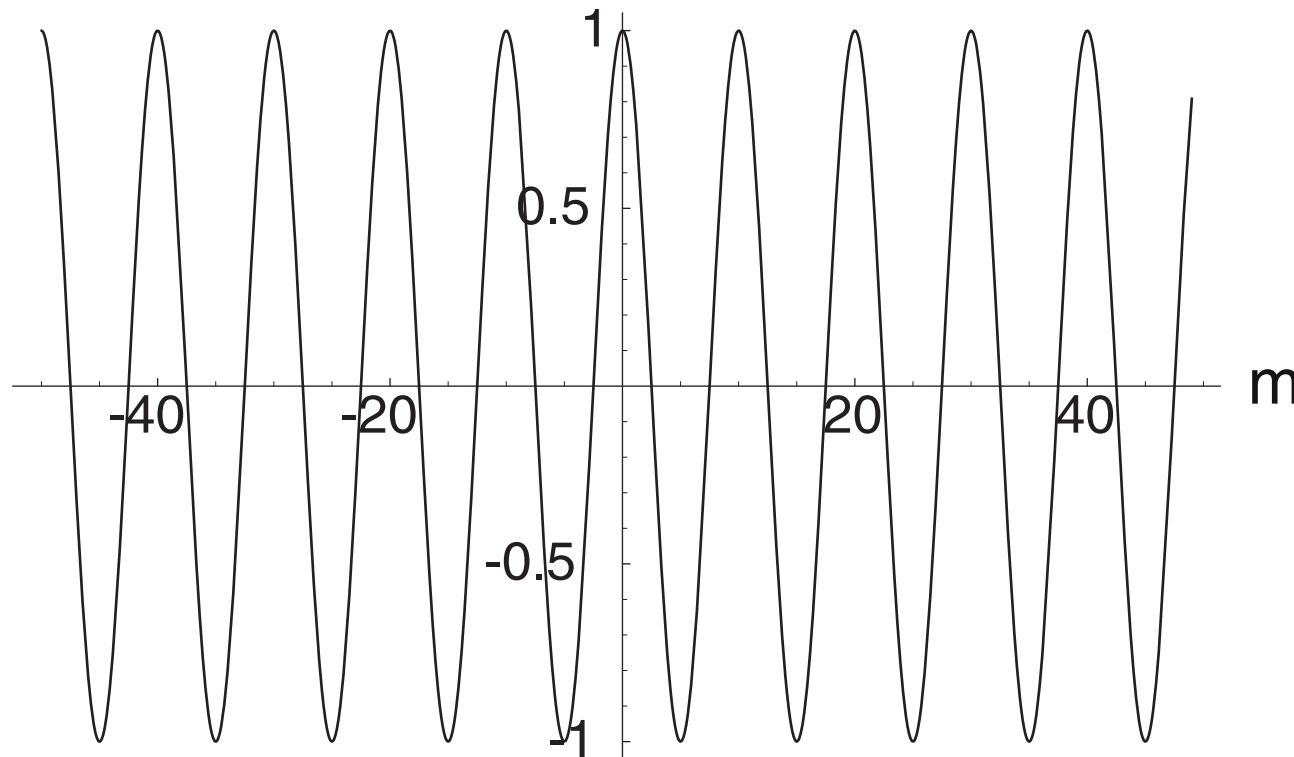
Real part of components of \mathbf{u}_1



SAMPLED FOURIER BASIS COMPONENTS (2)

- Values of $\text{Re} [u_n(t_m)] = e^{j\omega_n t_m}$ for $N = 50$, $n = 10$ and $m \in [-N, N - 1]$

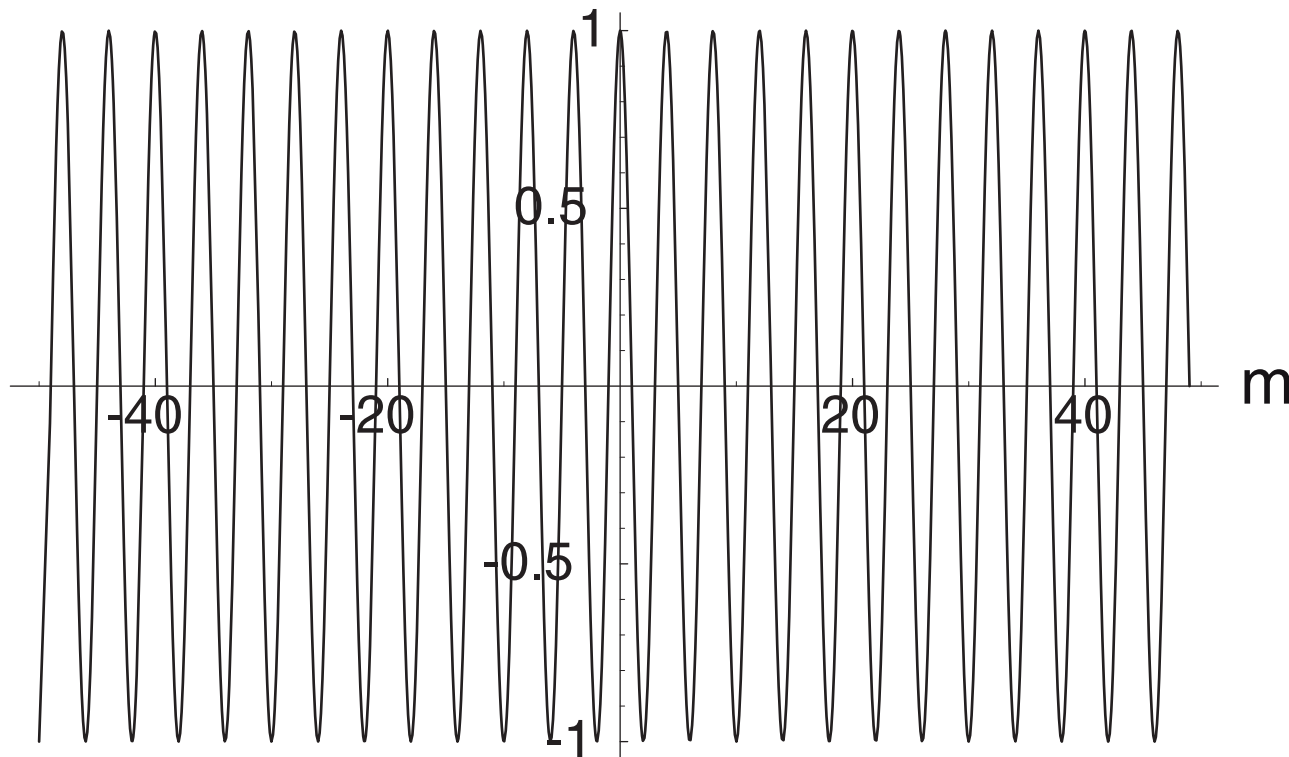
Real part of components of \mathbf{u}_{10}



SAMPLED FOURIER BASIS COMPONENTS (3)

- Values of $\text{Re} [u_n(t_m)] = e^{j\omega_n t_m}$ for $N = 50$, $n = 25$ and $m \in [-N, N - 1]$

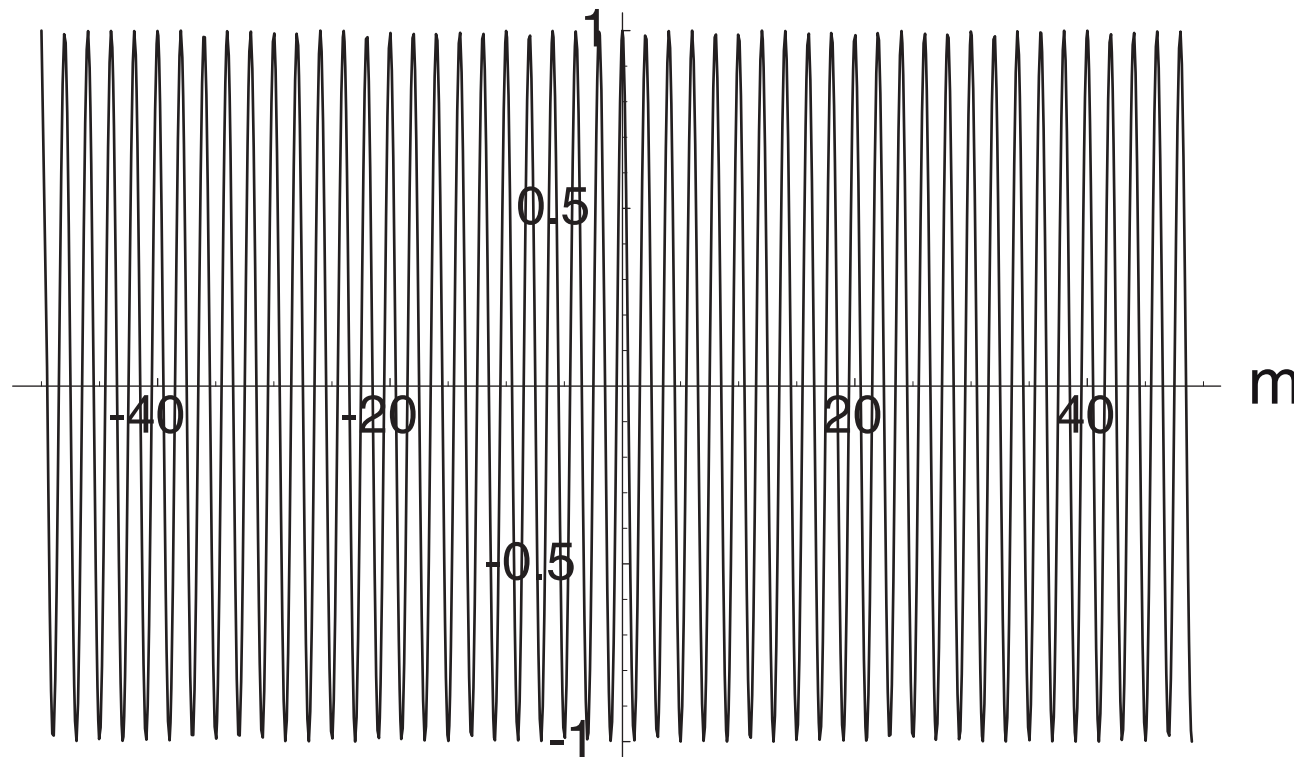
Real part of components of \mathbf{u}_{25}



SAMPLED FOURIER BASIS COMPONENTS (4)

- Values of $\text{Re} [u_n(t_m)] = e^{j\omega_n t_m}$ for $N = 50$, $n = 50$ and $m \in [-N, N - 1]$

Real part of components of \mathbf{u}_{50}



ALIASING (4)

- When a function is sampled, all frequencies outside the Nyquist interval

$$-\frac{1}{2}\omega_N \leq \omega_n < \frac{1}{2}\omega_N$$

are mapped (“aliased”) into the Nyquist interval

- ▷ Discrete Fourier series for a sampled function:

$$f(t_l) = \sum_{n=-N}^{N-1} C_n e^{j\omega_n t_l}$$

- ▷ Relation between Fourier coefficients for continuous vs. discrete Fourier series, showing where aliasing comes from mathematically:

$$C_m = \frac{1}{2N} \sum_{l=-N}^{N-1} f(t_l) e^{-j\omega_m t_l} = \frac{1}{2N} \sum_{l=-N}^{N-1} \sum_{n=-\infty}^{\infty} c_n e^{j(\omega_n - \omega_m) t_l}$$

$$C_m = \sum_{q=-\infty}^{\infty} c_{m+2Nq}$$

ALIASING (5)

- Calculation of the discrete Fourier coefficients using the discrete orthogonality relations:

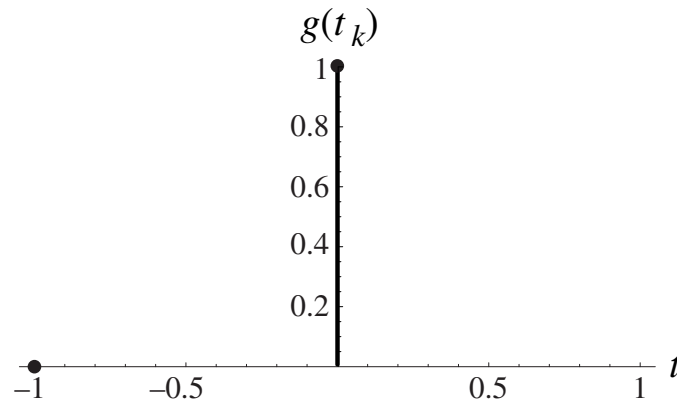
$$\begin{aligned} C_m &= \frac{1}{2N} \sum_{l=-N}^{N-1} f(t_l) e^{-j\omega_m t_l} \\ &= \frac{1}{2N} \sum_{l=-N}^{N-1} \sum_{n=-\infty}^{\infty} c_n e^{j(\omega_n - \omega_m) t_l} \\ \Rightarrow C_m &= \sum_{q=-\infty}^{\infty} c_{m+2Nq} \end{aligned}$$

where

$$-N \leq m \leq N - 1$$

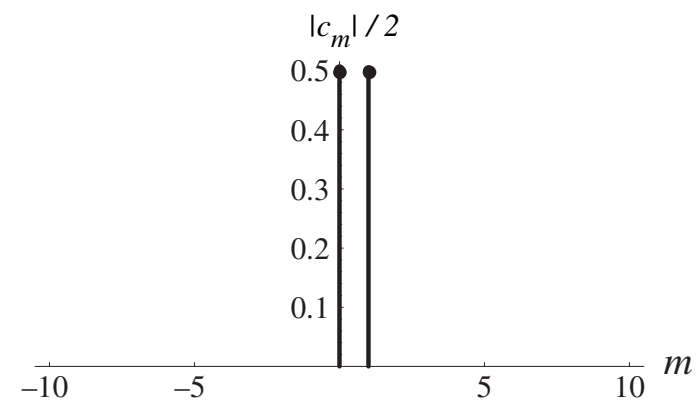
COARSELY SAMPLED SQUARE WAVE AND ITS DFT

Sampled square wave



(a)

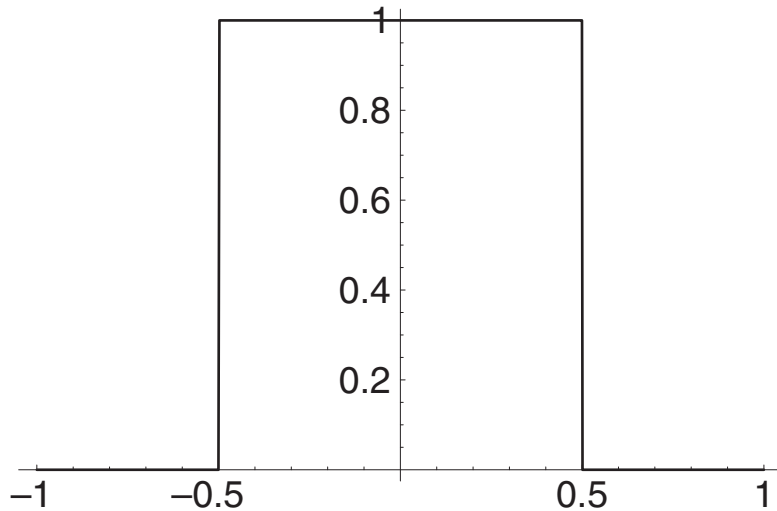
Discrete Fourier transform



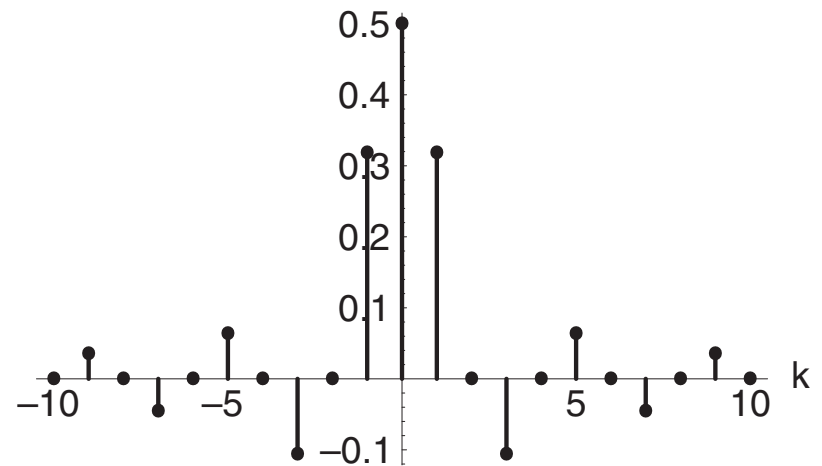
(b)

CONTINUOUS SQUARE WAVE AND ITS FOURIER COEFFICIENTS

Square wave



Fourier coefficients



ALIASING (6)

- Aliasing is a property of all expansions in orthogonal functions that are sampled on a discrete set

▷ Example: Let

$$f(t) = \sum_{n=0}^{\infty} c_n P_n(t) \text{ for all } x \in [-1, 1]$$

($P_n = n^{\text{th}}$ -order Legendre polynomial). On $\{t_0 = -1, t_1 = 1\}$,

$$f(t_l) = C_0 P_0(t_l) + C_1 P_1(t_l)$$

where $P_0(t) = 1$, $P_1(t) = x$, and

$$C_m = \sum_{q=0}^{\infty} c_{m+2q} \quad (m = 0, 1)$$

Sampling \Rightarrow the Legendre expansion is aliased, not truncated.