Numerical Inversion of Laplace Transforms Using a Fourier Series Approximation

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ABSTRACT. A method is presented for numerically inverting a Laplace transform that requires, in addition to the transform function itself, only sine, cosine, and exponential functions. The method is conceptually much like the method of Dubner and Abate, which approximates the inverse function by means of a Fourier cosine series. The method presented here, however, differs from theirs in two important respects. First of all, the Fourier series contains additional terms involving the sine function selected such that the error in the approximation is less than that of Dubner and Abate and such that the Fourier series approximates the inverse function on an interval of twice the length of the corresponding interval in Dubner and Abate's method. Second, there is incorporated into the method in this paper a transformation of the approximating series into one that converges very rapidly. In test problems using the method it has routinely been possible to evaluate inverse transforms with considerable accuracy over a wide range of values of the independent variable using a relatively few determinations of the Laplace transform itself.

KEY WORDS AND PHRASES: Laplace transform, Fourier series, inversion, numerical integration

CR CATEGORIES: 3.12, 3.24, 5.12, 5.17, 5.18, 5.19

1. *Introduction*

The Laplace transform of a real-valued function $f(t)$, $t \geq 0$, is defined as

$$F(s) = \int_0^\infty e^{-st} f(t) dt. \tag{1}$$

Throughout, we shall assume that $f(t)$ is piecewise continuous and of exponential order α (i.e. $|f(t)| \leq Me^{\alpha t}$), in which case the transform function $F(s)$ is defined for $\mathrm{Re}(s) > \alpha$. Starting with $F(s)$, the inverse transform $f(t)$ is given by the well-known inversion formula

$$f(t) = (1/2\pi i) \int_{a-i\infty}^{a+i\infty} e^{st} F(s)\, ds,$$

or, alternatively,

$$f(t) = (e^{at}/\pi) \int_0^\infty [\mathrm{Re}\,\{F(s)\} \cos \omega t - \mathrm{Im}\,\{F(s)\} \sin \omega t]\, d\omega, \tag{2}$$

$s = a + i\omega$, where a can be any real number greater than α. Equations (1) and (2) can also be replaced by the cosine transform pair

$$\mathrm{Re}\,\{F(s)\} = \int_0^\infty e^{-at} f(t) \cos \omega t\, dt, \tag{3}$$

$$f(t) = (2e^{at}/\pi) \int_0^\infty \mathrm{Re}\,\{F(s)\} \cos \omega t\, d\omega, \tag{4}$$

or by the sine transform pair

$$\mathrm{Im}\,\{F(s)\} = -\int_0^\infty e^{-at} f(t) \sin \omega t\, dt, \tag{5}$$

$$f(t) = (-2e^{at}/\pi) \int_0^\infty \mathrm{Im}\,\{F(s)\} \sin \omega t\, d\omega. \tag{6}$$

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The technique proposed by Dubner and Abate [1] for numerically inverting $F(s)$ is essentially a trapezoidal rule approximation to (4) which involves only Re $\{F(s)\}$. The method is conceptually simple and easy to program on a digital computer. An essential feature of the method is that an expression for the error in the computed inverse transform is available which allows one to control the maximum error in the inversion technique. Specifically, Dubner and Abate show that $f(t) = f_c(t) - E_c$ where $f_c(t)$ is the proposed approximation to $f(t)$ defined by

$$f_c(t) = (2e^{at}/T)\left[\frac{1}{2}F(a) + \sum_{k=1}^{\infty} \text{Re}\,\{F(a + k\pi i/T)\}\cos(k\pi t/T)\right] \tag{7}$$

with the error

$$E_c = \sum_{n=1}^{\infty} \exp(-2nat)\,\{f(2nT + t) + \exp(2at)f(2nT - t)\}, \tag{8}$$

where the parameters a, T satisfy the conditions $T > t$ and $a > \alpha$. Although Dubner and Abate assume $\alpha \leq 0$ and restrict a to be positive, this is not essential to their derivation (see [2]). Using the bound $|f(t)| \leq Me^{\alpha t}$ in (8) and summing the resulting geometric series, we find that

$$E_c \leq Me^{\alpha t}[\exp\{2(a - \alpha)t\} + 1]/[\exp\{2(a - \alpha)T\} - 1]. \tag{9}$$

By choosing $a - \alpha$ sufficiently large with $T > t$, the error E_c can be made as small as desired.

One shortcoming of the method is that usually the series (7) converges slowly. One must often sum hundreds of terms before convergence to three significant figures is attained. A modification of the method has been proposed in [3] that mitigates this difficulty. If one takes $T = 2t$ in (7), then

$$f_c(t) = (e^{at}/t)\left[\frac{1}{2}F(a) + \sum_{k=1}^{\infty} \text{Re}\,\{F(a + k\pi i/t)\}(-1)^k\right]. \tag{10}$$

This series can be summed more quickly than (7) since there are no cosines to compute. Furthermore, Simon, Stroot, and Weiss [3] demonstrate that when this alternating form of the series is used, a Euler transformation significantly increases the rate at which the series converges.On the other hand, when (10) is used, the argument at which the transform is computed now depends upon t and the transform $F(s)$ must now be computed for a different set of s-values for each distinct t. For those applications which occur frequently in practice in which the numerical computation of $F(s)$ itself is quite time-consuming, it may not be economical to use (10).

From the various inversion formulas (2), (4), and (6) one would surmise that similar numerical inversion techniques could be developed that utilize Im $\{F(s)\}$ rather than Re $\{F(s)\}$, or perhaps utilize a combination of both the real and imaginary parts. After all, in order to compute Re $\{F(s)\}$ one would normally simultaneously compute Im $\{F(s)\}$, and somehow this additional available information should be put to use. In Section 2 we extend the inversion method of Dubner and Abate to a method that utilizes these already computed imaginary parts of $F(s)$. As it turns out, the series involved in our method can be made to converge more rapidly than (7) and does not share the aforementioned drawback of (10).

2. *Derivation*

The same general argument used in [1] will be followed. Essentially, we will obtain the Fourier series for a function $g_0(t)$ that is periodic with period $2T$ and equal to $f(t)e^{-at}$ on the interval $(0, 2T)$. It turns out that the Fourier coefficients of the series may be approximated using $F(s)$. In order that the Fourier series converge to $g_0(t)$ at points of discontinuity we shall impose the condition that $f(t) = \{f(t+) + f(t-)\}/2$ for all t at which

$f(t)$ is discontinuous. For $n = 0, 1, 2, \cdots$, define $g_n(t)$, $-\infty < t < \infty$, by $g_n(t) = f(t)e^{-at}$, $2nT \le t < 2(n+1)T$, with $g_n(t)$ specified elsewhere from the condition that it be periodic with period $2T$. The Fourier series representation of each $g_n(t)$ is given by

$$g_n(t) = \tfrac{1}{2}A_{n,0} + \sum_{k=1}^{\infty} \{A_{n,k}\cos(k\pi t/T) + B_{n,k}\sin(k\pi t/T)\}, \qquad (11)$$

where the Fourier coefficients are

$A_{n,k} = (1/T)\int_{2nT}^{2(n+1)T} e^{-at}f(t)\cos(k\pi t/T)dt$, $B_{n,k} = (1/T)\int_{2nT}^{2(n+1)T} e^{-at}f(t)\sin(k\pi t/T)dt$.

By summing (11) with respect to n and noting that

$$\int_0^\infty e^{-at}f(t)\cos(k\pi t/T)dt = \operatorname{Re}\{F(a + ik\pi/T)\},$$
$$\int_0^\infty e^{-at}f(t)\sin(k\pi t/T)dt = -\operatorname{Im}\{F(a + ik\pi/T)\},$$

we find that

$$\sum_{n=0}^{\infty} g_n(t) = (1/T)\left[\frac{1}{2}F(a) + \sum_{k=1}^{\infty}\{\operatorname{Re}\{F(a + k\pi i/T)\}\cos(k\pi t/T) - \operatorname{Im}\{F(a + k\pi i/T)\}\sin(k\pi t/T)\}\right]. \qquad (12)$$

Now on the interval $(0, 2T)$, $g_0(t) = f(t)e^{-at}$, so from (12) we obtain the approximation $\tilde{f}(t)$ to the inverse transform given by

$$\tilde{f}(t) = (e^{at}/T)\left[\frac{1}{2}F(a) + \sum_{k=1}^{\infty}\{\operatorname{Re}\{F(a + k\pi i/T)\}\cos(k\pi t/T) - \operatorname{Im}\{F(a + k\pi i/T)\}\sin(k\pi t/T)\}\right], \qquad (13)$$

where $f(t) = \tilde{f}(t) - E$,

$$E = e^{at}\sum_{n=1}^{\infty} g_n(t) = e^{at}\sum_{n=1}^{\infty}\exp\{-a(2nT + t)\}f(2nT + t) = \sum_{n=1}^{\infty} e^{-2naT}f(2nT + t). \qquad (14)$$

Expression (13) is the desired approximation formula for $f(t)$. We note that it is simply a trapezoidal rule approximation to (2), just as (7) is a similar approximation to (4). We also note that (13) contains all of the terms in (7) plus additional terms involving $\operatorname{Im}\{F(s)\}$. By taking one half of each side of (13) and subtracting it from (7), then multiplying the result by 2, we get immediately an inversion procedure using only $\operatorname{Im}\{F(s)\}$, namely $f(t) = f_s(t) - E_s$ where

$$f_s(t) = (-2e^{at}/T)\sum_{k=1}^{\infty}\operatorname{Im}\{F(a + k\pi i/T)\}\sin(k\pi t/T) \qquad (15)$$

and

$$E_s = \sum_{n=1}^{\infty} e^{-2nat}\{f(2nT + t) - e^{2at}f(2nT - t)\}. \qquad (16)$$

In fact, the approximate inverse $\tilde{f}(t)$ is simply the average $\{f_c(t) + f_s(t)\}/2$ of the approximate inverses using only cosine terms and sine terms, respectively. However, by comparing (8), (14), and (16) we see that the error E is also the average $(E_c + E_s)/2$. Furthermore a sizable portion of the error drops out when E_c and E_s are averaged, so that usually E will be significantly smaller than either E_c or E_s. Moreover, the parameter a can be chosen so that $\tilde{f}(t)$ approximates $f(t)$ in the interval $(0, 2T)$ whereas the error when using $f_c(t)$ or $f_s(t)$ can be made small only for t in the interval $(0, T)$.

Error Analysis. There are two sources of error in the method, excluding roundoff error. First, we have the error term E due to the fact that the Fourier coefficients used in the Fourier series for $g_0(t)$ are not exact but are only approximations obtained using $F(s)$. Second, since the series in (13) is not summed to infinity, there is truncation error.

Using our assumption that $|f(t)| \leq Me^{\alpha t}$ we can sum (14) to get

$$E \leq Me^{\alpha t}/(e^{2T(a-\alpha)} - 1), \quad 0 < t < 2T. \tag{17}$$

It follows that by choosing a sufficiently larger than α we can make E as small as desired. In applying the technique we shall usually want convergence to at least 2 significant figures, so we shall require the relative error $E_R \equiv E/Me^{\alpha t} \leq .005$, which means that we can for all practical purposes replace (17) by

$$E \leq Me^{\alpha t}e^{-2T(a-\alpha)}, \quad 0 < t < 2T. \tag{18}$$

The error for $t = 0$ must be analyzed separately since usually there will be a discontinuity in the Fourier series representation for $g_b(t)$ at $t = 0$. At $t = 0$ the Fourier series for $g_b(t)$ converges to $g_b(0) = \{f(0) + f(2T)e^{-2aT}\}/2$. Hence from (13) and (14) we find that

$$\begin{aligned} \tilde{f}(0) - f(0)/2 &= E + f(2T)\exp(-2aT)/2 \\ &\leq M \exp\{-2T(a - \alpha)\} + M \exp\{-2T(a - \alpha)\}/2 = 3E/2. \end{aligned}$$

It follows that at $t = 0$ the method approximates $f(0)/2$ rather than $f(0)$ with an error bound 50 percent greater than the error bound for $t > 0$.

The error bound (18) provides for a simple algorithm for computing $f(t)$ to a predetermined accuracy. Suppose the numerical value of $f(t)$ is desired over a range of t-values, of which the largest is t_{max}, and the relative error is to be no greater than E'. First T is chosen so that $2T > t_{max}$ and then (18) is used to compute a. Explicitly, we choose

$$a = \alpha - Ln(E')/2T. \tag{19}$$

The series (13) is then summed until it has converged to the desired number of significant figures. It should also be pointed out that, if necessary, the parameter α can be computed from the transform $F(s)$. One simply takes α to be a number slightly larger than $\max\{\mathrm{Re}(P); P \text{ is a pole of } F(s)\}$.

Usually it is possible to increase the rate of convergence of (13) and thereby reduce truncation error by using a suitable series transformation. We have considered two transformations, the Euler Transformation (ET), which was used by Simon, Stroot, and Weiss [3], and the epsilon algorithm (EPAL) [4]. In all test problems considered to date, the EPAL has proved to be superior to the ET in speeding convergence of (13). The EPAL has without exception significantly improved the rate of convergence, and it is highly recommended that the EPAL be used in conjunction with the inversion method. The EPAL may be briefly described as follows. Suppose we wish to approximate the sum of the series $\sum_{n=1}^{\infty} a_n$ using the $2N + 1$ partial sums $S_m = \sum_{n=1}^{m} a_n$, $m = 1, 2, \cdots, 2N + 1$. We define $\epsilon_{p+1}^{(m)} = \epsilon_{p-1}^{(m+1)} + [\epsilon_p^{(m+1)} - \epsilon_p^{(m)}]^{-1}$ with $\epsilon_0^{(m)} = 0$, $\epsilon_1^{(m)} = S_m$. Then the sequence $\epsilon_1^{(1)}, \epsilon_3^{(1)}, \epsilon_5^{(1)}, \cdots, \epsilon_{2N+1}^{(1)}$ is a sequence of successive approximations to the sum of the series that will often better approximate the sum than the untransformed sequence $S_1, \cdots, S_{2N+1}$.

One can predict the local convergence properties of the series from the behavior of Fourier series. One would expect convergence to be slower in the neighborhood of a discontinuity of $f(t)$ due to the Gibbs phenomena, and this turns out to be the case. In addition, the method artifically introduces a discontinuity at $t = 0$ which tends to slow convergence in the neighborhood of the origin, although this does not usually appear to be very serious.

3. *Examples and Comments on Implementation*

From the examples of this section one can get some idea of the performance of the numerical inversion formula (13) with different choices of the parameters E_R, T, and N (N is the number of terms used in the sum). We shall also compare our method (13) with (7) and (10), which have been considered by earlier authors. All of the calculations reported here were performed in double precision on an IBM 370-Model 145 digital computer.

We shall invert the following four transforms:

$$F_1(s) = (s-1)/[(s-1)^2+1] - 1/s; \quad f_1(t) = e^t \cos t - 1;$$

$$F_2(s) = 2/s - 1/(s+1); \quad f_2(t) = 2 - e^{-t};$$

$$F_3(s) = (2/s - 1/(s+1))e^{-5s}; \quad f_3(t) = \begin{cases} 0, & t < 5, \\ 2 - \exp(-t+5), & t > 5; \end{cases}$$

$$f_4(s) = (s_2 + s + 1)^{-1}; \quad f_4(t) = (2/\sqrt{3}) \exp(-t/2) \sin(t\sqrt{3}/2).$$

We note that $f_1(t)$ is oscillatory and unbounded whereas $f_2(t)$ is bounded and monotone increasing. The third function $f_3(t)$ is obtained from $f_2(t)$ by shifting $f_2(t)$ five time units to the right, leaving a step discontinuity in $f_3(t)$ at $t = 5$. The function f_4 is a damped oscillatory function and was used as a test problem in [1] and [3].

In Table I we compare the performance of the ET and EPAL on our series. We note that the untransformed series converges slowly, particularly for the larger t-values. The Euler transformed series converges more quickly than the untransformed series, but the EPAL gives uniformly more accuracy than either of the other two methods. With the EPAL, 29 terms are sufficient to achieve an accuracy of 6–8 significant figures except at $t = 0, 1$. It should be emphasized that to achieve this accuracy the EPAL must be applied to (13) without rearranging the terms. We noted earlier that (13) is simply the average of (7) and (15). However, the EPAL being a nonlinear transformation, if it is applied to (7) and (15) separately and then the results are averaged, considerable accuracy is lost.

From Table II one gets some idea of the effect on the rate of convergence of different choices for the parameters E_R and T. In general, we see that smaller values of T improve convergence for small t but have the opposite effect for large t. Convergence is usually best for t near T. We have already seen that we must take $2T > t_{max}$ in order to control E_R, and from Table II we see that if $2T$ is too near t_{max} then convergence may be slowed drastically at the larger t-values. It appears that in this example the better choice for T is perhaps 7.5 with $E_R = 10^{-8}$ and 6 for $E_R = 10^{-4}$. Generally, the smaller E_R is, the smaller T can be without adversely affecting convergence for large t-values. As a rule of

TABLE I. COMPARISON OF UNTRANSFORMED SERIES AND ET AND EPAL TRANSFORMATIONS IN EVALUATING $f_1(t) = e^t \cos t - 1$

Equation (13) was used with $E_R = 10^{-8}$, $T = 7.5$, $\alpha = 0$, and $N = 29$. Results for EPAL are exact to number of significant digits recorded.

t	Untransformed	ET	EPAL (exact)
0	1.03	1.05	1.00
1	1.469 1.456 (N=31)	1.469 1.455 (N=31)	1.468 1.468 (N=31)
2	-3.063	-3.064	-3.07493
3	-1.970	-1.9881	-1.988453
4	-33.6	-35.68765	-35.687732
5	64.3	42.09904	42.09920
6	301.5	387.338	387.36034
8	3×10^5	-436.6	-433.7295
10	-2×10^7	-18481.777	-18481.780

TABLE II. EFFECT OF VARYING PARAMETERS ON EVALUATING $f_1(t) = e^t \cos t - 1$ USING EPAL

Numbers in table are: significant figure of error—digits of error. Example: 5—2 signifies an error of 2 digits in 5th significant digit.

		$E_R=10^{-4}$			$E_R=10^{-8}$		
t	N	T=5	T=6	T=7.5	T=6	T=7.5	T=10
0	15	2-1	2-1	2-1	3-8	3-8	2-2
	29	3-2	3-3	3-7	3-3	3-3	2-1
1	15	4-1	5-5	3-1	4-1	3-2	3-2
	29	5-1	5-3	5-5	5-2	5-2	5-1
2	15	4-2	4-5	4-5	4-5	3-2	5-6
	29	5-6	5-1	5-2	7-1	7-6	5-1
3	15	5-2	4-2	5-3	4-1	4-5	3-1
	29	5-2	5-1	5-2	9-2	8-3	7-1
4	15	5-1	5-5	4-1	5-4	6-2	3-1
	29	5-1	5-5	5-5	8-1	9-6	8-3
5	15	4-1	5-4	4-1	6-3	6-6	3-2
	29	4-1	5-4	5-6	9-6	8-4	6-1
6	15	4-2	5-3	5-2	9-5	7-4	4-4
	29	4-2	5-3	5-2	9-5	8-1	9-2
8	15	3-7	4-1	4-2	4-6	6-2	4-1
	29	4-2	4-1	4-2	8-1	8-5	8-2
10	15	*	4-1	3-4	1-10	6-2	5-2
	29	*	5-2	5-2	4-6	9-2	

*series did not converge

thumb, we have found that for $10^{-2} \le E_R \le 10^{-8}$, the value $T = .80t_{max}$ gives fairly optimal results.

We note that with $E = 10^{-4}$ the series has essentially converged to within the tolerance specified by E_R with 15 terms in the series, as additional terms do not significantly improve the accuracy. We also note that our error bound is fairly tight since with $E_R = 10^{-4}$ the series does not converge to within more than 4 significant figures of the correct answer.

Comparing columns I and III of Table III, we observe that when the EPAL is used, the method using both $\mathrm{Re}\{F(s)\}$ and $\mathrm{Im}\{F(s)\}$ (eq. (13)) converges more rapidly than eq. (7) which utilizes only $\mathrm{Re}\{F(s)\}$. In fact, of the possibilities of using (7), (13), or (15) either untransformed or in conjunction with ET or EPAL, the combination of (13) with EPAL has given the best results in all test problems thus far. (Results using (15) are similar to those using (7) and therefore were not recorded.)

In column IV we have used (10) in conjunction with EPAL. We note that eq. (10) cannot be used at $t = 0$. However, at all other t-values use of (10) with EPAL gave an accuracy of six significant figures with only 13 terms in the sum. If the inverse transform is needed for only a few t-values, one should consider using eq. (10) with EPAL.

As pointed out earlier the method used in column I of Table III will converge at $t = 0$ to $f(0)/2$ rather than $f(0)$, so that a jump discontinuity is introduced artificially at $t = 0$. In the method reported in column II, this jump was removed by inverting $F_2^*(s) = F_2(s) - f(0)/s$ and taking $f_2(t) = f_2^*(t) + f(0)$. As can be seen, this transformation improves convergence somewhat and is recommended in those situations occurring frequently in practice in which $f(0)$ is known beforehand.

From the values of the computed inverse recorded in the first column of Table IV, we observe the perturbation predicted earlier in the neighborhood of the discontinuity at

TABLE III. EVALUATION OF $f_2(t) = 2 - \exp(-t)$

Parameter values: I and II, $E_R = 10^{-6}$, $T = 3.75$; III, $E_R = 10^{-6}$, $T = 10$ ($2aT = 13.8$); IV, $E_R = 10^{-6}$ ($2at = 13.8$).

		I	II	III	IV	
t	N	Using (13) with EPAL	Using (13) with EPAL & removing jump at t=0	Using (7) with EPAL	Using (10) with EPAL	Exact
0	13 31	.494 .498	1.009 1.002	.985 .996	—	1.0
1	13 31	1.634 1.63212	1.6328 1.63212	1.67 1.633	1.63212	1.63212
2	13 31	1.86461	1.86468	1.862 1.86467	1.86466	1.86466
3	13	1.95022	1.95022	1.975 1.9509	1.95021	1.95021
4	13 31	1.98169	1.98169	1.9803 1.98168	1.98169	1.98168
5	13	1.99326	1.99327	1.99326	1.99326	1.99326

TABLE IV. EVALUATION OF $f_3(t)$

Parameter values are $E_R = 10^{-6}$, $T = 6.5$, $N = 13$.

t	Using (13) with EPAL: regular	Using (13) with EPAL: jump removed	Exact
0	.000002	.000001	0
1	.000002	.000001	0
2	.000002	.000001	0
3	.000007	.000003	0
4	.0018	.0004	0
5	.4990	1.01	1.0
6	1.623 1.637(N=15)	1.76 1.630(N=15)	1.63212
7	1.8651	1.8657	1.86466
8	1.9508	1.9510	1.95021
9	1.9805	1.98168	1.98168
10	1.9928	1.99325	1.99326

$t = 5$ due to the Gibbs phenomenon. In computing the results of column II this discontinuity was removed by inverting $F_3^*(s) = F_3(s) - e^{-5s}/s$ and then taking $f_3(t) = f_3^*(t)$, $t < 5$, and $f_3(t) = f_3^*(t) + 1$, $t \geq 5$. As seen from the table, this procedure generally hastened convergence. If when applying the method to a discontinuous function the location and height of the jumps are known beforehand, then they should be removed in this fashion. Even after removal of the jump we note that convergence is still relatively slower, in the neighborhood of $t = 5$, due to the nonexistence of the first derivative. However, from other examples not shown here, we conclude that the nonexistence of second or higher order derivatives has no noticeable affect upon the rate of convergence.

As a final example, in Table V, we compare our method against numerical results quoted in [1, 3]. Dubner and Abate's computations were made using (7) directly. Simon,

TABLE V. ACCURACY OF THREE NUMERICAL INVERSIONS OF $F_4(s)$
The values obtained using eq. (13) were computed with $E_R = 10^{-8}$, $T = 7.5$, $\alpha = -.5$.

t	$f_4(t)$ (exact)	Dubner and Abate [1] (N=2000)	Absolute Error Simon, Stroot and Weiss [3]	Using eq. (13) with EPAL (N=19)
1	.333507195	1×10^{-5}	5×10^{-5}(N=13)	5×10^{-4}
2	.419279630	2×10^{-5}	2×10^{-5}(N=13)	4×10^{-5}
3	.133242644	4×10^{-5}	4×10^{-6}(N=13)	3×10^{-7}
4	-.049529880	3×10^{-5}	0 (N=13)	2×10^{-6}
5	-.087942073	4×10^{-5}	2×10^{-6}(N=13)	2×10^{-6}
6	-.050892318	8×10^{-6}	1×10^{-6}(N=14)	1×10^{-7}
7	-.007643714	4×10^{-5}	5×10^{-6}(N=15)	6×10^{-9}
8	.012715096	2×10^{-5}	1×10^{-6}(N=15)	6×10^{-9}
9	.012804671	1×10^{-5}	1×10^{-6}(N=15)	3×10^{-9}
10	.005385481	2×10^{-5}	1×10^{-6}(N=16)	7×10^{-8}
Total no. of values of $F_4(s)$ needed		2000	140	19

Stroot, and Weiss used (10) with a Euler transformation. Overall, our method gives more accurate results than either of the other two. One should bear in mind that our results are based upon only 19 evaluations of the transform $F_4(s)$ versus 2000 evaluations using Dubner and Abate's method and 140 evaluations using the method of Simon, Stroot, and Weiss.

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