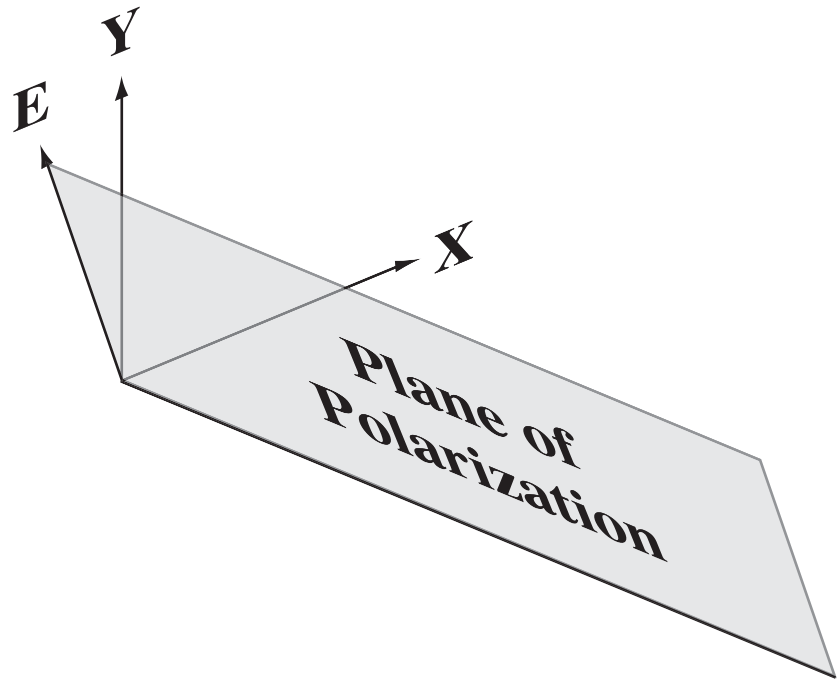
POLARIZATION

- The **state of polarization** (SOP) is the relation between the transverse components of **E**
  - ▷ Assume a plane wave propagating in the $+z$ direction
  - ▷ **Linear (plane) polarization**: E_y/E_x is real and independent of time
    - ∘ The **E** field vector lies in a fixed plane that includes the direction of propagation (see figure on next page)
  - ▷ **Circular polarization**:
    - ∘ The tip of the **E** vector describes a circle in the $x - y$ plane
    - ∘ In **right circular polarization**, **E** rotates *clockwise* looking toward the source (looking in the $-z$ direction)
  - ▷ **Elliptical polarization**:
    - ∘ The tip of the **E** vector describes an ellipse in the $x - y$ plane

PLANE OF POLARIZATION



- The direction of the electric field and the direction of propagation determine the **plane of polarization**

PLANE POLARIZATION

- The state of polarization of a plane wave is determined by the value of the relative phase angle, δ, and the ratio E_1/E_2, where

$$E_x = E_1 \cos(\omega t - \beta z), \qquad E_y = E_2 \cos(\omega t - \beta z + \delta)$$

- **Plane polarization** (also called **linear polarization**):

$$\delta = 0$$

  ▷ Real components of the **E** field of a plane wave:

$$E_x = E_1 \cos(\omega t - \beta z), \qquad E_y = E_2 \cos(\omega t - \beta z)$$

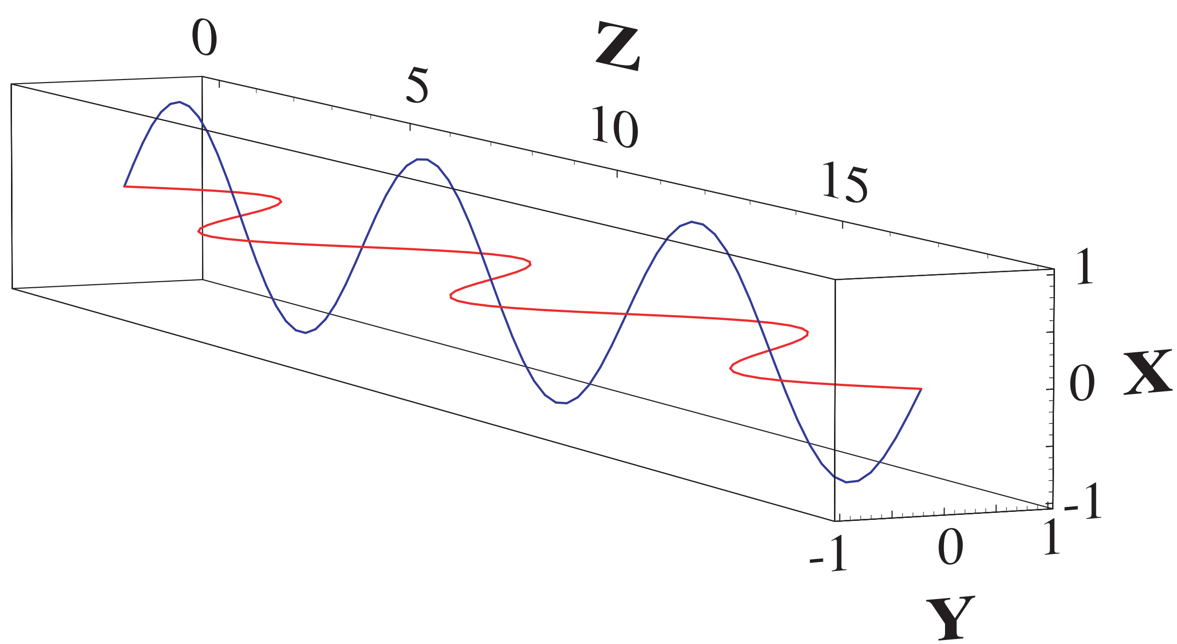
  ▷ The **E** vector makes an angle

$$\phi = \tan^{-1}(E_2/E_1)$$

  with the x axis

- Note that the word "plane" has been used with 2 different meanings on this slide!

SNAPSHOT OF PLANE-POLARIZED PLANE WAVE



- Red curve: Tip of the **E** vector Blue curve: Tip of the **H** vector
  The plane of polarization is the $x - z$ plane

CIRCULAR POLARIZATION

- **Circular polarization:**

$$\delta = \pm\frac{\pi}{2}, \qquad E_2 = E_1$$

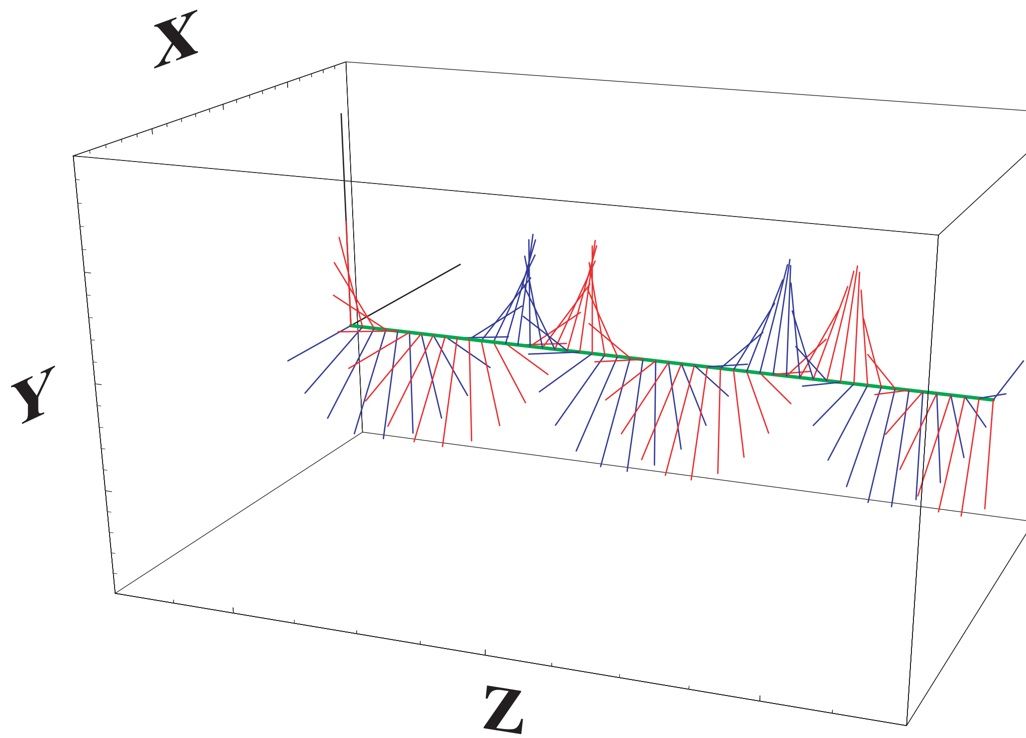
▷ Real components of **E**:

$$E_x = E_1 \cos(\omega t - \beta z), \qquad E_y = \mp E_1 \sin(\omega t - \beta z)$$

▷ For fixed z, the **E** vector describes a circle in the $x - y$ plane

- $\delta = \frac{\pi}{2}$ ($-$ sign on E_y): Right circular polarization
- $\delta = -\frac{\pi}{2}$ ($+$ sign on E_y): Left circular polarization
- The terms "right" and "left" circular polarization are confusing, because a right-circularly-polarized wave actually describes a left-handed screw as the wave propagates in z
  - ◇ For right-circularly-polarized light, the tip of the **E** vector rotates clockwise in the $x - y$ plane, from the point of view of someone looking toward the source (in the $-z$ direction)

SNAPSHOT OF CIRCULARLY-POLARIZED PLANE WAVE



- Red lines: **E** vectors   Blue lines: **H** vectors
  The light is left-circularly-polarized

ELLIPTICAL POLARIZATION

- **Elliptical polarization:**

$$E_2 \neq E_1, \quad \textbf{or} \quad E_2 = E_1 \quad \text{and} \quad \delta \neq \pm\frac{\pi}{2} \quad \text{and} \quad \delta \neq 0$$

▷ Real components of $\mathbf{E}$:

$$E_x = E_1 \cos(\omega t - \beta z), \qquad E_y = E_2 \cos(\omega t - \beta z + \delta)$$

▷ For fixed z, the $\mathbf{E}$ vector describes an ellipse in the $x - y$ plane

- Eliminating $\zeta = \omega t - \beta z$ from the equations for E_x and E_y results in

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 - 2\cos\delta \frac{E_x}{E_1}\frac{E_y}{E_2} = \sin^2\delta$$

which is the equation of an ellipse that makes an angle ψ with the x axis, where

$$\tan 2\psi = \frac{2E_1 E_2 \cos\delta}{E_1^2 - E_2^2}$$

ORTHOGONAL POLARIZATIONS (1)

- For a plane wave, there are always 2 orthogonal states of polarization
  - ▷ Simplest case: Orthogonal linear polarizations described by unit coordinate vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$
    - ◦ Orthogonality: $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$
    - ◦ Normalization: $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1 = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}}$
    - ◦ Example of orthogonally polarized fields:

$$\mathbf{E}_1 = \hat{\mathbf{x}} E_1 \cos(\omega t - \beta z), \qquad \mathbf{E}_2 = \hat{\mathbf{y}} E_2 \cos(\omega t - \beta z + \delta)$$

  - ▷ The important point: **Waves with orthogonal polarizations do not interfere with one another**
- Example: In a "single-mode" fiber, there are really two modes, one for each of the two orthogonal polarizations
  - ▷ These modes don't have the same group velocity
  - ▷ The difference in group velocities leads to polarization-mode dispersion, pulse spreading, and bandwidth limitations

ORTHOGONAL POLARIZATIONS (2)

- Orthogonal states of polarization in general
  - ▷ The state of polarization of the elliptically polarized field

$$\mathbf{E} = \mathrm{Re}\left[\left(\hat{\mathbf{x}} E_1 + \hat{\mathbf{y}} E_2 e^{j\delta}\right) e^{j(\omega t - \beta z)}\right] \qquad (E_1,\ E_2 \text{ real})$$

    is described by the complex unit vector

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{E_1^2 + E_2^2}} \left(\hat{\mathbf{x}} E_1 + \hat{\mathbf{y}} E_2 e^{j\delta}\right)$$

  - ▷ The orthogonal polarization is described by the complex unit vector

$$\hat{\mathbf{e}}_2 = \hat{\mathbf{z}} \times \hat{\mathbf{e}}_1^* = \frac{1}{\sqrt{E_1^2 + E_2^2}} \left(-\hat{\mathbf{x}} E_2 e^{-j\delta} + \hat{\mathbf{y}} E_1\right)$$

    - ∘ Orthogonality: $\hat{\mathbf{e}}_1^* \cdot \hat{\mathbf{e}}_2 = 0 = \hat{\mathbf{e}}_2^* \cdot \hat{\mathbf{e}}_1$
    - ∘ Normalization: $\hat{\mathbf{e}}_1^* \cdot \hat{\mathbf{e}}_1 = 1 = \hat{\mathbf{e}}_2^* \cdot \hat{\mathbf{e}}_2$
    - ∘ If $\psi \in (0, \pi)$, then $\hat{\mathbf{e}}_1$ describes a right elliptical polarization state, and $\hat{\mathbf{e}}_2$ describes a left elliptical polarization state

ORTHOGONAL POLARIZATIONS (3)

- Example of orthogonal states of circular polarization
  - ▷ A right-circularly-polarized wave is described by the unit vector

$$\hat{\mathbf{e}}_R = \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + j\hat{\mathbf{y}})$$

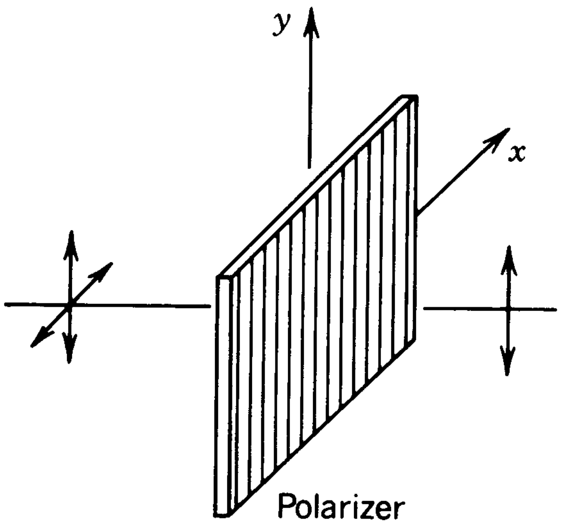
  - ▷ The orthogonal polarization is left-circularly-polarized and is described by the complex unit vector

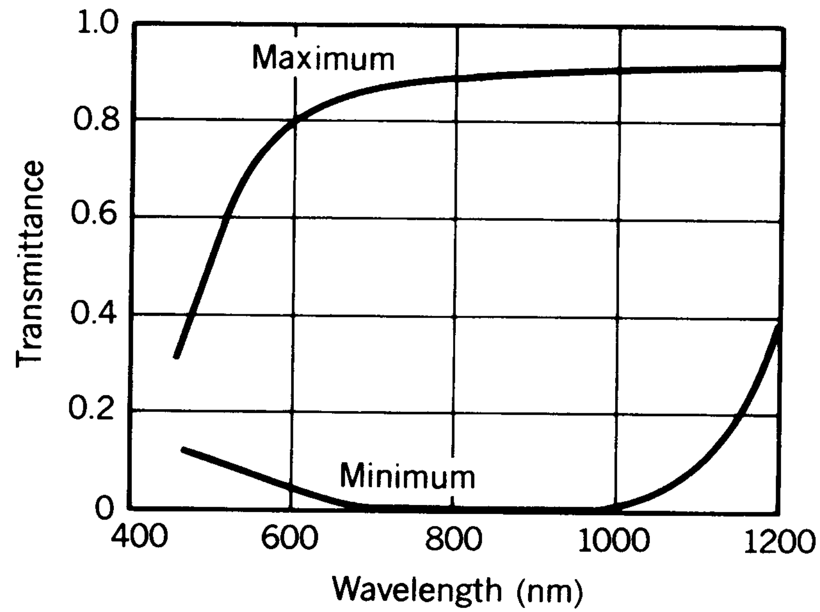
$$\hat{\mathbf{e}}_L = \frac{1}{\sqrt{2}} (j\hat{\mathbf{x}} + \hat{\mathbf{y}}) = \frac{j}{\sqrt{2}} (\hat{\mathbf{x}} - j\hat{\mathbf{y}})$$

POLARIZERS AND ANALYZERS

- A **polarizer** prepares a specific state of polarization
- An **analyzer** blocks a specific state of polarization and transmits the orthogonal polarization
- Materials and techniques for making polarizers and analyzers:
  - ▷ Polarization-selective absorption
    - ∘ Some materials are **dichroic**
      - ◇ One of two states of polarization is absorbed more than the other
      - ◇ Example: Polaroid (as in sunglasses)
  - ▷ Brewster's-angle polarizers and analyzers
    - ∘ Problem: Brewster's angle is wavelength-dependent
  - ▷ Polarizing prisms
    - ∘ Best rejection ratio for the blocked polarization state
  - ▷ Wire-grid polarizers and analyzers
    - ∘ Used in the far infrared (where there are few birefringent materials)

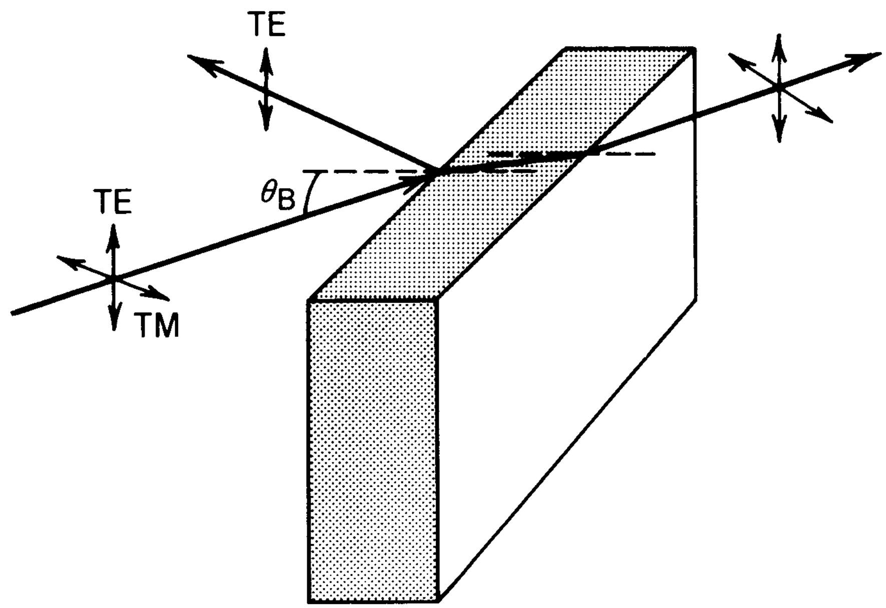
DICHROIC POLARIZERS





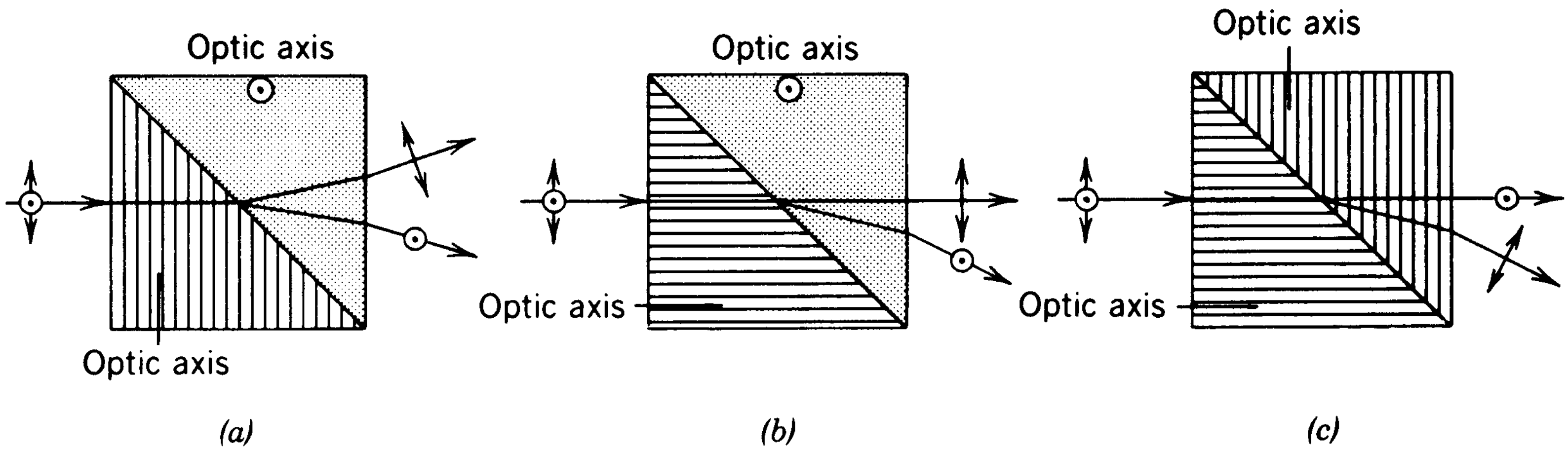
B. E. A. Saleh and M. C. Teich, *Fundamentals of Photonics* (Wiley, 1991), Fig. 6-6.1

BREWSTER'S-ANGLE POLARIZERS



B. E. A. Saleh and M. C. Teich, *Fundamentals of Photonics* (Wiley, 1991), Fig. 6-6.2

POLARIZING PRISMS



(a) Wollaston prism (b) Rochon prism (c) Sénarmont prism

B. E. A. Saleh and M. C. Teich, *Fundamentals of Photonics* (Wiley, 1991), Fig. 6-6.3

JONES VECTOR

- Complex electric field components (assuming propagation in the z direction):

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\,\mathrm{Re}\left[\mathcal{E}_x(z,t)e^{j(\omega t-\beta z)}\right] + \hat{\mathbf{y}}\,\mathrm{Re}\left[\mathcal{E}_y(z,t)e^{j(\omega t-\beta z)}\right]$$

$$\mathbf{E}(z,t) = \mathrm{Re}\left[\boldsymbol{\mathcal{E}}(z,t)e^{j(\omega t-\beta z)}\right]$$

▷ The time-averaged power per unit area is proportional to

$$\overline{\mathbf{E}\cdot\mathbf{E}} = \tfrac{1}{2}\left(|\mathcal{E}_x|^2 + |\mathcal{E}_y|^2\right)$$

- The **Jones vector** $\hat{\mathbf{s}}$ is a normalized, 2-component, complex vector, the components of which are proportional to $\mathcal{E}_x$ and $\mathcal{E}_y$:

$$\hat{\mathbf{s}}(z,t) = \begin{pmatrix} s_x(z,t) \\ s_y(z,t) \end{pmatrix} = \frac{1}{\left[|\mathcal{E}_x|^2 + |\mathcal{E}_y|^2\right]^{1/2}} \begin{pmatrix} \mathcal{E}_x(z,t) \\ \mathcal{E}_y(z,t) \end{pmatrix}$$

▷ Normalization of $\hat{\mathbf{s}}$:

$$\hat{\mathbf{s}}^\dagger\hat{\mathbf{s}} = |s_x|^2 + |s_y|^2 = 1$$

POLARIZATION MATRIX (JONES MATRIX)

- Modification of the state of polarization of narrowband light
  - ▷ General polarization-sensitive optical system:

$$\boldsymbol{\mathcal{E}}_{\text{out}} = \begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix} = \begin{pmatrix} \ell_{xx} & \ell_{xy} \\ \ell_{yx} & \ell_{yy} \end{pmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \mathbf{L}\boldsymbol{\mathcal{E}}_{\text{in}}$$

    - ∘ $\mathbf{L}$ is the **polarization matrix** (or **Jones matrix**)
  - ▷ Rotation of the plane of polarization from $\hat{\mathbf{x}}$ to $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$:

$$\mathbf{L} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

  - ▷ Polarizer/analyzer at angle α to the x axis:

$$\mathbf{L} = \begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix}$$

  - ▷ Phase plate (δ = retardation $= ckd[v_x^{-1} - v_y^{-1}] = (\beta_x - \beta_y)d$):

$$\mathbf{L} = \begin{pmatrix} e^{-j\beta_x d} & 0 \\ 0 & e^{-j\beta_y d} \end{pmatrix} = e^{-j\beta_x d}\begin{pmatrix} 1 & 0 \\ 0 & e^{j\delta} \end{pmatrix}$$

FULL vs. PARTIAL POLARIZATION

- If a beam of narrowband light is fully polarized, then there exist a polarization matrix $\mathbf{L}$ and an orientation of an analyzer such that the analyzer completely extinguishes the beam (zero transmitted intensity)
  - ▷ The Jones vector and Jones matrix describe fully polarized light
- Experimentally, for some beams of light, there is no orientation of an analyzer (or analyzer plus phase plate) that will achieve complete extinction
  - ▷ Such light is called **partially polarized**
  - ▷ The mathematical description of partially polarized light is based on optical coherence theory
    - ∘ Coherence is the ability to form interference fringes
    - ∘ The visibility of an interference pattern is (Michelson, 1891)

$$\mathcal{V} := \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$$

    - ∘ $\mathcal{V}$ can be related to correlation functions of the envelope of the electric field

PARTIAL POLARIZATION (1)

- How to measure the coherence of different components of $\mathbf{E}$:
  - ▷ Propagate $\mathbf{E} = \mathrm{Re}\left[\boldsymbol{\mathcal{E}} e^{j(\omega t - \beta z)}\right]$ through a birefringent phase plate so that E_y is retarded by a phase δ with respect to E_x
  - ▷ Component of $\boldsymbol{\mathcal{E}}$ along a direction $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$:

$$\mathcal{E}_n = \boldsymbol{\mathcal{E}} \cdot \hat{\mathbf{n}} = \mathcal{E}_x \cos\theta + \mathcal{E}_y\, e^{j\delta} \sin\theta$$

  - ▷ Average intensity of $\mathbf{E}$ field component along $\hat{\mathbf{n}}$:

$$\langle |\mathcal{E}_n|^2 \rangle = J_{xx} \cos^2\theta + J_{yy} \sin^2\theta + (J_{xy} e^{-j\delta} + J_{yx} e^{j\delta}) \cos\theta \sin\theta$$

    - ○ $\mathbf{J}$ is the **coherency matrix**

$$\mathbf{J} = \begin{pmatrix} \langle \mathcal{E}_x^* \mathcal{E}_x \rangle & \langle \mathcal{E}_x \mathcal{E}_y^* \rangle \\ \langle \mathcal{E}_x^* \mathcal{E}_y \rangle & \langle \mathcal{E}_y^* \mathcal{E}_y \rangle \end{pmatrix} = \left\langle \boldsymbol{\mathcal{E}} \boldsymbol{\mathcal{E}}^\dagger \right\rangle$$

    - ○ Note that $J_{yx} = J_{xy}^*$
    - ○ The correlation function $\langle \mathcal{E}_x \mathcal{E}_y^* \rangle$ is the ensemble average of $\mathcal{E}_x \mathcal{E}_y^*$

PARTIAL POLARIZATION (2)

- Coherency matrix $\mathbf{J}$ for important special cases:
  - ▷ Unpolarized light:

$$\mathbf{J} = \tfrac{1}{2}|\boldsymbol{\mathcal{E}}|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  - ▷ Plane-polarized light along direction $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$:

$$\mathbf{J} = |\boldsymbol{\mathcal{E}}|^2 \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

  - ▷ Circularly polarized light:
    - ∘ Assume space-time dependence $\boldsymbol{\mathcal{E}}\, e^{j(\omega t - \beta z)}$:

$$\mathbf{J} = \tfrac{1}{2}|\boldsymbol{\mathcal{E}}|^2 \begin{pmatrix} 1 & \mp j \\ \pm j & 1 \end{pmatrix}$$

    - ∘ Upper signs for right circular polarization
    - ∘ Lower signs for left circular polarization

- Total intensity $= |\boldsymbol{\mathcal{E}}|^2 = J_{xx} + J_{yy} = \text{trace}\,[\mathbf{J}]$

PARTIAL POLARIZATION (3)

- Modification of the coherency matrix $\mathbf{J}$ by a general polarization-sensitive optical system:

$$\mathbf{J}' = \left\langle \mathcal{E}'\mathcal{E}'^{\dagger} \right\rangle = \langle \mathbf{L}\mathcal{E}(\mathbf{L}\mathcal{E})^{\dagger} \rangle = \left\langle \mathbf{L}\mathcal{E}\mathcal{E}^{\dagger}\mathbf{L}^{\dagger} \right\rangle = \mathbf{L} \left\langle \mathcal{E}\mathcal{E}^{\dagger} \right\rangle \mathbf{L}^{\dagger} = \mathbf{L}\mathbf{J}\mathbf{L}^{\dagger}$$

  - ▷ This is a unitary similarity transformation, which doesn't change the trace or determinant
  - ▷ Therefore the trace and determinant of the coherency matrix are invariant after passage through a general lossless polarization-sensitive system:

$$\text{trace}\,[\mathbf{J}'] = \text{trace}\,[\mathbf{J}] \quad \text{(preservation of the total intensity)}$$

$$\det\,[\mathbf{J}'] = \det\,[\mathbf{J}] \quad \text{(preservation of the degree of polarization; see below)}$$

PARTIAL POLARIZATION (4)

- To measure the elements of the coherency matrix $\mathbf{J}$, make the following measurements:
  - (Average intensity of $\mathbf{E}$ field component along $\hat{\mathbf{x}}$) $= J_{xx}$
  - (Average intensity of $\mathbf{E}$ field component along $\hat{\mathbf{y}}$) $= J_{yy}$
  - Using a half-wave plate ($\delta = \pi$) as on a previous slide, measure (Average intensity of $\mathbf{E}$ field component along an axis at 45° to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$) $= \frac{1}{2}[J_{xx} + J_{yy} - (J_{xy} + J_{yx})]$
  - Using a quarter-wave plate ($\delta = \pi/2$) as on a previous slide, measure (Average intensity of $\mathbf{E}$ field component along an axis at 45° to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$) $= \frac{1}{2}[J_{xx} + J_{yy} - j(J_{xy} - J_{yx})]$

PARTIAL POLARIZATION (5)

- Desirable properties for a general definition of the degree of polarization, P:
  - ▷ $0 \leq P \leq 1$
  - ▷ $P = 0$ for completely unpolarized light
  - ▷ $P = 1$ for completely polarized light, regardless of the details of the state of polarization (plane, circular, or elliptical)
  - ▷ In terms of the coherency matrix, such a **degree of polarization** is

$$P = \left[1 - \frac{4\det\left[\mathbf{J}\right]}{(J_{xx} + J_{yy})^2}\right]^{\frac{1}{2}}$$

    - ○ Later, we will give a definition of the degree of polarization that is consistent with the definition of fringe visibility and results in the same formula for P

PARTIAL POLARIZATION (6)

- Examples of the degree of polarization:
  - ▷ Light fully plane-polarized along x:

$$\mathbf{J} = \begin{pmatrix} \langle|\mathcal{E}_x|^2\rangle & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \det[\mathbf{J}] = 0 \Rightarrow P = 1$$

  - ▷ Completely unpolarized light, with no correlation between E_x and E_y, and with equal intensities in the x and y components:

$$\mathbf{J} = \begin{pmatrix} \langle|\mathcal{E}_x|^2\rangle & 0 \\ 0 & \langle|\mathcal{E}_x|^2\rangle \end{pmatrix} \Rightarrow \det[\mathbf{J}] = \left(\langle|\mathcal{E}_x|^2\rangle\right)^2$$

$$(J_{xx} + J_{yy})^2 = \left(2\langle|\mathcal{E}_x|^2\rangle\right)^2 = 4\left(\langle|\mathcal{E}_x|^2\rangle\right)^2$$

$$\Rightarrow P = 0$$

PARTIAL POLARIZATION (7)

- General relations between the degree of polarization and $\mathbf{J}$:
  - ▷ Light is fully polarized if & only if $P = 1$, and

$$\boxed{P = 1 \Leftrightarrow \det[\mathbf{J}] = 0}$$

  - ▷ Light is completely unpolarized if & only if $P = 0$, and

$$P = 0 \Leftrightarrow 4\det[\mathbf{J}] = (J_{xx} + J_{yy})^2$$

$$\Leftrightarrow 4(J_{xx}J_{yy} - J_{xy}J_{yx}) = J_{xx}^2 + 2J_{xx}J_{yy} + J_{yy}^2$$

$$\Leftrightarrow -4|J_{xy}|^2 = (J_{xx} - J_{yy})^2$$

Since $|J_{xy}|^2 \geq 0$ and $(J_{xx} - J_{yy})^2 \geq 0$, the last equation can hold if & only if

$$|J_{xy}|^2 = 0 \quad \text{and} \quad (J_{xx} - J_{yy})^2 = 0$$

$$\boxed{P = 0 \Leftrightarrow \mathbf{J} = \begin{pmatrix} \langle|\mathcal{E}_x|^2\rangle & 0 \\ 0 & \langle|\mathcal{E}_x|^2\rangle \end{pmatrix}}$$

PARTIAL POLARIZATION (8)

- Properties of the coherency matrix $\mathbf{J}$:
  - ▷ J_{xx} and J_{yy} are power densities; hence $J_{xx} \geq 0$, $J_{yy} \geq 0$
  - ▷ $\det[\mathbf{J}] \geq 0$; hence $J_{xx}J_{yy} \geq J_{xy}J_{yx} \Rightarrow J_{xx}J_{yy} \geq |J_{xy}|^2$
  - ▷ $\mathbf{J}$ is Hermitian, $\mathbf{J}^\dagger = \mathbf{J}$, and therefore:
    - ∘ The eigenvalues λ_1, λ_2 of $\mathbf{J}$ are real
    - ∘ There exists a unitary matrix $\mathbf{U}$ such that

$$\mathbf{U}\mathbf{J}\mathbf{U}^\dagger = \mathbf{J}_{\text{diag}} := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

    - ∘ $\mathbf{U}$ is the polarization matrix of some optical system
  - ▷ $\mathbf{J}_{\text{diag}}$ is equal to the coherency matrix of unpolarized light, plus the coherency matrix of plane-polarized light:

$$\mathbf{J}_{\text{diag}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & 0 \end{pmatrix}$$

    - ∘ This assumes (without loss of generality) that $\lambda_1 \geq \lambda_2$

PARTIAL POLARIZATION (9)

- $\mathbf{J}_{\text{diag}}$ is equal to the coherency matrix of unpolarized light, plus the coherency matrix of plane-polarized light:

$$\mathbf{J}_{\text{diag}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & 0 \end{pmatrix}$$

  ▷ This assumes (without loss of generality) that $\lambda_1 \geq \lambda_2$

- The **degree of polarization** is the ratio of the intensity of the polarized part to the total intensity:

$$P := \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$$

  ▷ Consistent with the definition of the visibility of interference fringes

  ▷ Degree of polarization in terms of the original coherency matrix:

$$P = \left[\frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}\right]^{\frac{1}{2}} = \left[\frac{(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2}{(\lambda_1 + \lambda_2)^2}\right]^{\frac{1}{2}} = \left[1 - \frac{4\det[\mathbf{J}]}{(\text{trace}\,[\mathbf{J}])^2}\right]^{\frac{1}{2}}$$

STOKES VECTOR

- The **Stokes vector S** is a real vector, the 4 components of which are related to the components of the Jones vector $\hat{\mathbf{s}}$ and the elements of the coherency matrix **J** as follows:

$$\mathbf{S}(z,t) = \begin{pmatrix} S_0(z,t) \\ S_1(z,t) \\ S_2(z,t) \\ S_3(z,t) \end{pmatrix} = \begin{pmatrix} \left\langle |\mathcal{E}_x|^2 \right\rangle + \left\langle |\mathcal{E}_y|^2 \right\rangle \\ S_0 \left[\left\langle |s_x|^2 \right\rangle - \left\langle |s_y|^2 \right\rangle \right] \\ S_0 \left[\left\langle s_x s_y^* \right\rangle + \left\langle s_x^* s_y \right\rangle \right] \\ S_0 \left[j \left(\left\langle s_x s_y^* \right\rangle - \left\langle s_x^* s_y \right\rangle \right) \right] \end{pmatrix} = \begin{pmatrix} J_{xx} + J_{yy} \\ J_{xx} - J_{yy} \\ J_{xy} + J_{yx} \\ j\left(J_{xy} - J_{yx}\right) \end{pmatrix}$$

  - ▷ The components of **S** are called the **Stokes parameters**
  - ▷ The Stokes vector and the coherency matrix provide alternative but equivalent descriptions of partial polarization

STOKES VECTOR/COHERENCY MATRIX PROPERTIES (1)

- Determinant of the coherency matrix:

$$\det[\mathbf{J}] = J_{xx}J_{yy} - J_{xy}J_{yx} = J_{xx}J_{yy} - |J_{xy}|^2$$

- Trace of the coherency matrix:

$$\boxed{\operatorname{trace}[\mathbf{J}] = J_{xx} + J_{yy} = S_0}$$

- Quadratic relations involving the components of the Stokes vector **S**:

$$S_0^2 - S_1^2 = (J_{xx} + J_{yy})^2 - (J_{xx} - J_{yy})^2 = 4J_{xx}J_{yy}$$
$$S_2^2 + S_3^2 = (J_{xy} + J_{yx})^2 - (J_{xy} - J_{yx})^2 = 4J_{xy}J_{yx}$$

$$\boxed{S_0^2 - \left(S_1^2 + S_2^2 + S_3^2\right) = 4\det[\mathbf{J}]}$$

$$\boxed{S_0^2 - \left(S_1^2 + S_2^2 + S_3^2\right) = (1 - P^2)\left(\operatorname{trace}[\mathbf{J}]\right)^2}$$

  ▷ Only three of the Stokes parameters are independent

STOKES VECTOR/COHERENCY MATRIX PROPERTIES (2)

- Degree of polarization in terms of **S**:

$$
\begin{aligned}
P &= \left[1 - \frac{4\det[\mathbf{J}]}{(J_{xx}+J_{yy})^2}\right]^{\frac{1}{2}} \\
&= \left[1 - \frac{S_0^2 - (S_1^2 + S_2^2 + S_3^2)}{S_0^2}\right]^{\frac{1}{2}} \\
&= \frac{\left[S_1^2 + S_2^2 + S_3^2\right]^{\frac{1}{2}}}{S_0}
\end{aligned}
$$

$$
\boxed{S_1^2 + S_2^2 + S_3^2 = (PS_0)^2}
$$

STOKES VECTOR/COHERENCY MATRIX PROPERTIES (3)

- Fully polarized light:

$$P = 1 \Leftrightarrow \det[\mathbf{J}] = 0 \Leftrightarrow S_1^2 + S_2^2 + S_3^2 = S_0^2$$

  - ▷ Plane-polarized light in general:

$$J_{xy} \text{ real} \Leftrightarrow J_{yx} = J_{xy} \Leftrightarrow S_3 = 0$$

    - ∘ Plane-polarized along x: $J_{yy} = 0$ and $J_{xy} = J_{yx} = 0 \Rightarrow S_1 = S_0$
    - ∘ Plane-polarized along y: $J_{xx} = 0$ and $J_{xy} = J_{yx} = 0 \Rightarrow S_1 = -S_0$

  - ▷ Fully right-circularly-polarized light:

$$J_{xy} = -\tfrac{1}{2} j |\boldsymbol{\mathcal{E}}|^2 \Rightarrow S_3 = S_0$$

- Partially polarized light:

$$S_1^2 + S_2^2 + S_3^2 = (P S_0)^2$$

- Completely unpolarized light:

$$S_1^2 + S_2^2 + S_3^2 = 0$$

STOKES VECTOR/COHERENCY MATRIX PROPERTIES (4)

- Unitary norm of the Stokes vector $\mathbf{S}$:

$$
\begin{aligned}
\mathbf{S}^\dagger \mathbf{S} &= (J_{xx} + J_{yy})^2 + (J_{xx} - J_{yy})^2 + (J_{xy} + J_{yx})^2 - (J_{xy} - J_{yx})^2 \\
&= 2\left(J_{xx}^2 + J_{yy}^2 + 2J_{xy}J_{yx}\right) \\
&= 2\left((J_{xx} + J_{yy})^2 + 2\left(J_{xy}J_{yx} - J_{xx}J_{yy}\right)\right) \\
&= 2\left((\operatorname{trace}[\mathbf{J}])^2 - 2\det[\mathbf{J}]\right) \\
&= 2\,(\operatorname{trace}[\mathbf{J}])^2 \left(1 - 2\frac{\det[\mathbf{J}]}{(\operatorname{trace}[\mathbf{J}])^2}\right) \\
&= 2\,(\operatorname{trace}[\mathbf{J}])^2 \left(1 - \tfrac{1}{2}(1 - P^2)\right) \\
&= (\operatorname{trace}[\mathbf{J}])^2 \left(1 + P^2\right) \\
&= (\text{total intensity})^2 \left(1 + (\text{degree of polarization})^2\right)
\end{aligned}
$$

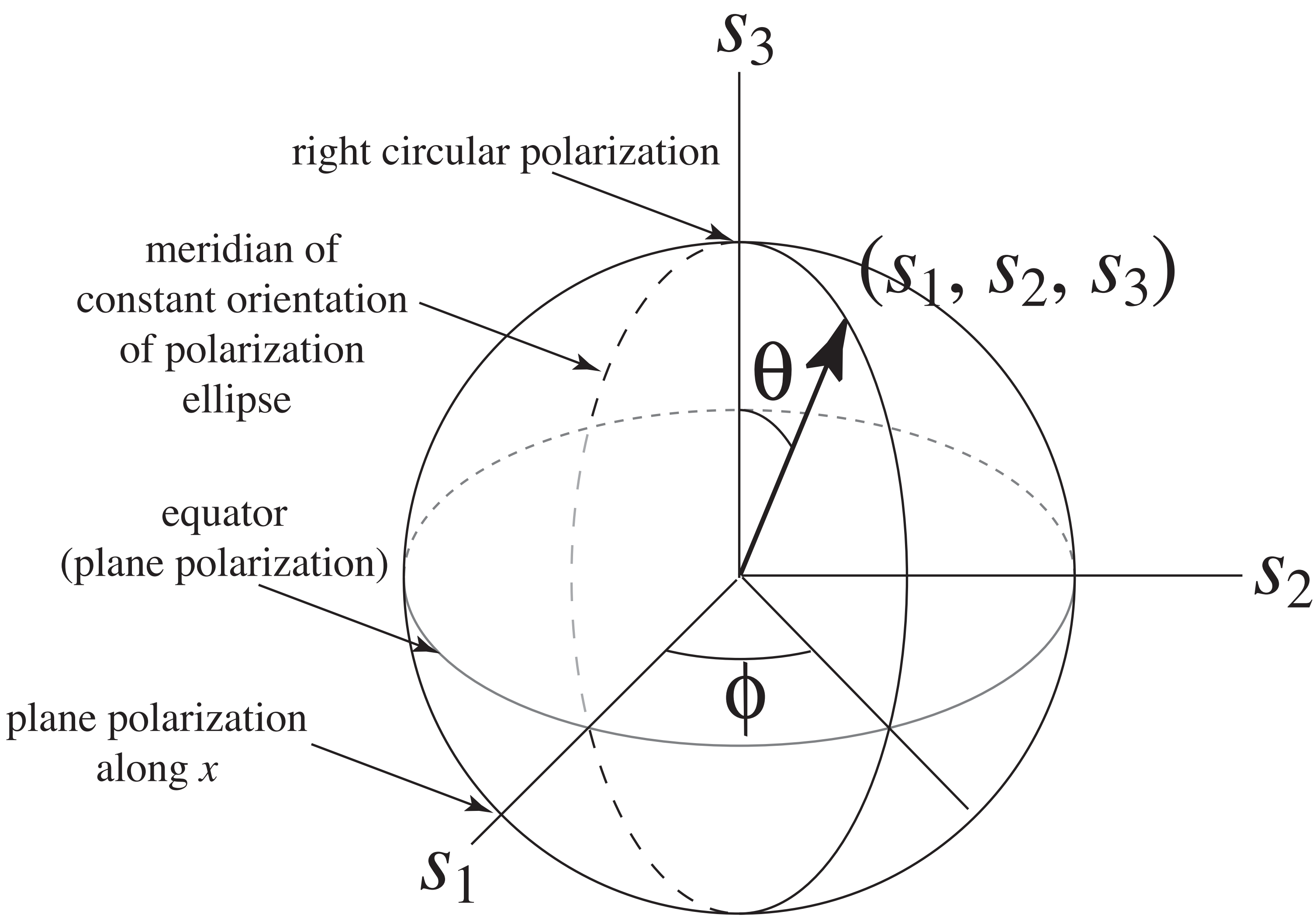
POINCARÉ SPHERE (1)

- The **Poincaré sphere** displays the three independent Stokes parameters S_1, S_2, S_3 as points on, or inside, a sphere
  - ▷ Two different versions are widely used to visualize states of polarization
  - ▷ The **full Poincaré sphere**, in which the point (S_1, S_2, S_3), which describes the state of polarization, lies at a distance P from the center of the sphere
    - ∘ Not particularly convenient for practical display
  - ▷ The **normalized Poincaré sphere**, which uses the **normalized Stokes parameters** (for $P > 0$)

$$s_1 = \frac{S_1}{PS_0}, \quad s_2 = \frac{S_2}{PS_0}, \quad s_3 = \frac{S_3}{PS_0}$$

    - ∘ The point (s_1, s_2, s_3) lies on the surface of a sphere with radius 1, since $s_1^2 + s_2^2 + s_3^2 = 1$
    - ∘ In effect, this displays the state of polarization of the polarized part of the beam

POINCARÉ SPHERE (2)



POINCARÉ SPHERE (3)

- Plane-polarized light in general:

$$s_3 = 0 \Rightarrow (s_1, s_2, s_3) \text{ lies on the equator}$$

  ▷ Plane-polarized along x: $s_1 = 1$, $s_2 = s_3 = 0$

  ▷ Plane-polarized along y: $s_1 = -1$, $s_2 = s_3 = 0$

- Fully right-circularly-polarized light: $s_3 = 1$, $s_1 = s_2 = 0$
- Right-hand elliptically polarized light: upper hemisphere

  ▷ The major axis of the ellipse makes an angle $\phi/2$ with the x axis

  ▷ The polar angle is related to the phase difference δ between $\mathcal{E}_y$ and $\mathcal{E}_x$:

$$\cos\theta = \frac{2|\mathcal{E}_x||\mathcal{E}_y|\sin\delta}{|\mathcal{E}_x|^2 + |\mathcal{E}_y|^2}$$