FIBER PARAXIAL WAVE EQUATION (9)

- General fiber paraxial wave equation:

$$\left[\frac{\partial}{\partial z'} + \left(i\frac{\beta_2}{2}\frac{\partial^2}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t'^3}\right)\right]\overline{\mathcal{E}}(z', t') = -\frac{\alpha}{2}\overline{\mathcal{E}} - \frac{2\pi i}{\beta_0 c^2}\frac{e^{-i(\beta_0 z - \omega_0 t)}\displaystyle\int_{\mathcal{A}} \psi_m(\mathbf{r}_T, \omega_0)^* \hat{\mathbf{e}} \cdot \frac{\partial^2 \overline{\mathbf{P}}_{NL}}{\partial t^2} d^2\mathbf{r}_T}{\displaystyle\int_{\mathcal{A}} |\psi_m(\mathbf{r}_T, \omega_0)|^2 \, d^2\mathbf{r}_T}$$

  - ▷ Group velocity dispersion effects: β_2, β_3
  - ▷ Nonlinear effects: $\overline{\mathbf{P}}_{NL}$
  - ▷ Mode volume effects: $\displaystyle\int_{\mathcal{A}} |\psi_m(\mathbf{r}_T, \omega_0)|^2 \, d^2\mathbf{r}_T$

ELECTRIC POLARIZATION (6)

- Evaluation of $\dfrac{\partial^2}{\partial t^2}\mathbf{P}_{NL}$:
  - ▷ Let
$$\mathbf{P}_{NL,m} = \mathrm{Re}\left[-i\hat{\mathbf{e}}_m \psi_n(\mathbf{r}_T, \omega_m)\mathcal{P}_m(z,\, t)e^{i[\beta(\omega_m)z - \omega_m t]}\right]$$
  - ▷ Second time derivative:
$$\frac{\partial^2}{\partial t^2}\mathbf{P}_{NL,m} = \mathrm{Re}\left\{-i\hat{\mathbf{e}}_m \psi_n(\mathbf{r}_T, \omega_m)e^{i[\beta(\omega_m)z - \omega_m t]}\left[-\omega_m^2 - 2i\omega_m\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\right]\mathcal{P}_m(z,\, t)\right\}$$
  - ▷ If $\mathcal{P}_m$ is, and remains, slowly varying on the time scale defined by an optical cycle, then $\partial\mathcal{P}_m/\partial t$ and $\partial^2\mathcal{P}_m/\partial t^2$ may be neglected

ELECTRIC POLARIZATION (7)

- Coupled-wave method for third-order nonlinear electric polarization:
  - ▷ Use a different field for each frequency and each orthogonal polarization:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4$$

  - ▷ Assume the envelope forms

$$\mathbf{E}_l = \mathrm{Re}\left[\hat{\mathbf{e}}_l \psi_l(\mathbf{r}_T, \omega_l)\mathcal{E}_l(z,\, t) e^{i[\beta(\omega_l)z - \omega_l t]}\right]$$

and

$$\mathbf{P}_{NL} = \sum_n \mathrm{Re}\left[-i\hat{\mathbf{e}}_n \psi_n(\mathbf{r}_T, \omega_n)\mathcal{P}_n(z,\, t) e^{i[\beta(\omega_n)z - \omega_n t]}\right]$$

  - ▷ Electric fields at frequencies ω_1, ω_2, ω_3 generate polarization components at frequencies

$$\pm\omega_k, \quad \pm 3\omega_k, \quad \pm(2\omega_k - \omega_l), \quad \pm(\omega_k + \omega_l - \omega_m) \quad (k, l, m = 1, 2, 3)$$

FIBER PARAXIAL WAVE EQUATION (10)

- Envelope form for the nonlinear polarization at ω_n:

$$\mathbf{P}_{NL,n} = \mathrm{Re}\left[-i\hat{\mathbf{e}}_n \psi_n(\mathbf{r}_T, \omega_n)\mathcal{P}_n(z,\, t)e^{i(\beta_n z - \omega_n t)}\right]$$

- Coupled-wave fiber paraxial wave equation:

$$\begin{aligned}\left[\frac{\partial}{\partial z'} + \left(i\frac{\beta_2}{2}\frac{\partial^2}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t'^3}\right)\right]\overline{\mathcal{E}}_n(z', t')\\ = -\frac{\alpha}{2}\overline{\mathcal{E}}_n + \frac{2\pi\mu_{nklm}}{\beta(\omega_n)c^2}\left(\omega_n^2 + 2i\omega_n\frac{\partial}{\partial t}\right)\overline{\mathcal{P}}_n(z', t')\end{aligned}$$

NONLINEAR POLARIZATION FOR FWM (1)

- Nonlinear polarization:

$$\mathbf{P}_{NL} = 4\chi^{(3)}(\mathbf{E}\cdot\mathbf{E})\mathbf{E} = \sum_n \mathrm{Re}\left[-i\hat{\mathbf{e}}_n\psi_n(\mathbf{r}_T)\mathcal{P}_{NL,n}(z,\,t)e^{i(\beta_n z-\omega_n t)}\right]$$

$$= \tfrac{1}{2}\sum_n \left[-i\hat{\mathbf{e}}_n\psi_n\mathcal{P}_{NL,n}e^{i(\beta_n z-\omega_n t)} + \text{c.c.}\right]$$

The sum on n runs over all the frequencies that occur in $(\mathbf{E}\cdot\mathbf{E})\mathbf{E}$

- Assume three fields, with possibly different frequencies and polarizations, and possibly in different modes:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = \sum_{l=1}^{3} \mathrm{Re}\left[\hat{\mathbf{e}}_l\psi_l(\mathbf{r}_T)\mathcal{E}_l(z,\,t)e^{i(\beta_l z-\omega_l t)}\right]$$

$$= \tfrac{1}{2}\sum_{l=1}^{3}\left[\hat{\mathbf{e}}_l\psi_l\mathcal{E}_l e^{i(\beta_l z-\omega_l t)} + \text{c.c.}\right]$$

NONLINEAR POLARIZATION FOR FWM (2)

- Expand $(\mathbf{E}\cdot\mathbf{E})\mathbf{E} = \mathbf{E}^2\mathbf{E}$:

$$
\begin{aligned}
&(\mathbf{E}\cdot\mathbf{E})\mathbf{E} \\
&\quad = \left(\mathbf{E}_1^2 + \mathbf{E}_2^2 + \mathbf{E}_3^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2 + 2\mathbf{E}_1\cdot\mathbf{E}_3 + 2\mathbf{E}_2\cdot\mathbf{E}_3\right)(\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3) \\
&\qquad = \left[\mathbf{E}_1^2\mathbf{E}_1 + \mathbf{E}_2^2\mathbf{E}_2 + \mathbf{E}_3^2\mathbf{E}_3\right] \\
&\qquad\quad + \left[\left(\mathbf{E}_2^2 + \mathbf{E}_3^2\right)\mathbf{E}_1 + \left(\mathbf{E}_1^2 + \mathbf{E}_3^2\right)\mathbf{E}_2 + \left(\mathbf{E}_1^2 + \mathbf{E}_2^2\right)\mathbf{E}_3\right. \\
&\qquad\quad + (2\mathbf{E}_1\cdot\mathbf{E}_2)\mathbf{E}_2 + (2\mathbf{E}_1\cdot\mathbf{E}_3)\mathbf{E}_3 + (2\mathbf{E}_1\cdot\mathbf{E}_2)\mathbf{E}_1 \\
&\qquad\quad \left. + (2\mathbf{E}_2\cdot\mathbf{E}_3)\mathbf{E}_3 + (2\mathbf{E}_1\cdot\mathbf{E}_3)\mathbf{E}_1 + (2\mathbf{E}_2\cdot\mathbf{E}_3)\mathbf{E}_2\right] \\
&\qquad\quad + \left[(2\mathbf{E}_1\cdot\mathbf{E}_2)\mathbf{E}_3 + (2\mathbf{E}_1\cdot\mathbf{E}_3)\mathbf{E}_2 + (2\mathbf{E}_2\cdot\mathbf{E}_3)\mathbf{E}_1\right]
\end{aligned}
$$

- Each group of square-bracketed terms predicts different nonlinear-optical phenomena

EVALUATION OF $\mathbf{E}_1^2\mathbf{E}_1$

- Envelope form:

$$\mathbf{E}_1 = \tfrac{1}{2}\hat{\mathbf{e}}_1 \left[W_1 + W_1^*\right]$$

where

$$W_1 = e^{i[\beta(\omega_1)z - \omega_1 t]}\psi_1(\mathbf{r}_T, \omega_1)\mathcal{E}_1(z, t)$$

- Then

$$\begin{aligned}\mathbf{E}_1^2\mathbf{E}_1 &= \tfrac{1}{8}\hat{\mathbf{e}}_1 \left[W_1 + W_1^*\right]\left[W_1 + W_1^*\right]^2 \\ &= \tfrac{1}{8}\hat{\mathbf{e}}_1 \left[W_1 + W_1^*\right]\left[W_1^2 + 2|W_1|^2 + W_1^{*2}\right] \\ &= \tfrac{1}{8}\hat{\mathbf{e}}_1 \left[(W_1^3 + W_1^{*3}) + (W_1^2 W_1^* + W_1^{*2}W_1) + 2|W_1|^2(W_1 + W_1^*)\right]\end{aligned}$$

  - ▷ Coefficient of third-harmonic terms at $\pm 3\omega_1$ is $\frac{1}{8}$
  - ▷ Coefficient of self-phase modulation terms at $\pm\omega_1$ is $\frac{3}{8}$

EVALUATION OF $\mathbf{E}_2^2\mathbf{E}_1$

- Envelope form:

$$\mathbf{E}_l = \tfrac{1}{2}\hat{\mathbf{e}}_l\left[W_l + W_l^*\right] \quad (l = 1, 2)$$

where

$$W_l = e^{i[\beta(\omega_l)z - \omega_l t]}\psi_l(\mathbf{r}_T, \omega_l)\mathcal{E}_l(z, t)$$

- Then

$$\begin{aligned}
\mathbf{E}_2^2\mathbf{E}_1 &= \tfrac{1}{8}\hat{\mathbf{e}}_1\left[W_1 + W_1^*\right]\left[W_2 + W_2^*\right]^2 \\
&= \tfrac{1}{8}\hat{\mathbf{e}}_1\left[W_1 + W_1^*\right]\left[W_2^2 + 2|W_2|^2 + W_2^{*2}\right] \\
&= \tfrac{1}{8}\hat{\mathbf{e}}_1\left[W_1W_2^2 + W_1^*W_2^{*2}\right. \\
&\quad \left. + 2|W_2|^2(W_1 + W_1^*) + W_1^*W_2^2 + W_1W_2^{*2}\right]
\end{aligned}$$

  - ▷ Coefficient of FWM terms at $\pm(2\omega_2 - \omega_1)$ is $\frac{1}{8}$
  - ▷ Coefficient of cross-phase modulation terms at $\pm\omega_1$ is $\frac{2}{8}$

EVALUATION OF $2(\mathbf{E}_1 \cdot \mathbf{E}_2)\mathbf{E}_2$

- Envelope form:

$$\mathbf{E}_l = \tfrac{1}{2}\hat{\mathbf{e}}_l \left[W_l + W_l^*\right] \quad (l = 1, 2)$$

where

$$W_l = e^{i[\beta(\omega_l)z - \omega_l t]}\psi_l(\mathbf{r}_T, \omega_l)\mathcal{E}_l(z, t)$$

- Then

$$\begin{aligned} 2(\mathbf{E}_1 \cdot \mathbf{E}_2)\mathbf{E}_2 &= \tfrac{2}{8}(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 \left[W_1 + W_1^*\right]\left[W_2 + W_2^*\right]^2 \\ &= \tfrac{2}{8}(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 \left[W_1 + W_1^*\right]\left[W_2^2 + 2|W_2|^2 + W_2^{*2}\right] \\ &= \tfrac{2}{8}(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 \big[W_1 W_2^2 + W_1^* W_2^{*2} \\ &\qquad + 2|W_2|^2(W_1 + W_1^*) + W_1^* W_2^2 + W_1 W_2^{*2}\big] \end{aligned}$$

  ▷ Coefficient of FWM terms at $\pm(2\omega_2 - \omega_1)$ is $\frac{2}{8}$

  ▷ Coefficient of cross-phase modulation terms at $\pm\omega_1$ is $\frac{4}{8}$

NONLINEAR POLARIZATION FOR FWM (3)

- After substitution of the envelope form of $\mathbf{E}_l$, 3 kinds of terms occur in $(\mathbf{E} \cdot \mathbf{E})\mathbf{E}$:
  - ▷ General form: $\mathcal{E}_l^3$ or c.c.
    - ◦ Coefficient (same polarization): $\frac{1}{8}$
    - ◦ Frequencies: $\pm 3\omega_l$
    - ◦ Third-harmonic generation
  - ▷ General form: $|\mathcal{E}_l|^2\mathcal{E}_l$, $\mathcal{E}_k^2\mathcal{E}_l^*$ or c.c.
    - ◦ Coefficient (same polarization): $\frac{3}{8}$
    - ◦ Frequencies: $\pm\omega_l$ or $\pm(2\omega_k \pm \omega_l)$
    - ◦ SPM, degenerate FWM
  - ▷ General form: $|\mathcal{E}_k|^2\mathcal{E}_l$, $\mathcal{E}_k^*\mathcal{E}_l\mathcal{E}_m$ or c.c.
    - ◦ Coefficient (same polarization): $\frac{6}{8}$
    - ◦ Frequencies: $\pm\omega_l$, $\pm(\omega_k + \omega_l \pm \omega_m)$
    - ◦ XPM, non-degenerate FWM

NONLINEAR POLARIZATION FOR FWM (4)

- Compact formula for $(\mathbf{E}\cdot\mathbf{E})\mathbf{E}$:

$$\mathbf{E}^2\mathbf{E} = \frac{1}{8}\sum_{k,l,m} D_{|k|,|l|,|m|}\psi_k\psi_l\psi_m\mathcal{E}_k\mathcal{E}_l\mathcal{E}_m\, e^{i\{[\beta(\omega_k)+\beta(\omega_l)+\beta(\omega_m)]z-(\omega_k+\omega_l+\omega_m)t\}}$$

$$\text{where} \qquad k,\ l,\ m = \pm 1,\ \pm 2,\ \pm 3,$$

$$\mathcal{E}_{-|k|} = \mathcal{E}^*_{|k|}, \quad \psi_{-|k|} = \psi^*_{|k|}, \quad \beta_{-|k|} = -\beta_{|k|}, \quad \omega_{-|k|} = -\omega_{|k|},$$

and, when all waves have the same polarization,

$$D_{|k|,|l|,|m|} = \begin{cases} 1, & \text{if } k = l = m; \\ 3, & \text{if } k = -l = m; \\ 6, & \text{if } \begin{cases} k = -l \neq m & \text{or if} \\ |k|, |l|, |m| & \text{are different.} \end{cases} \end{cases}$$

- Compact formula for $\mathbf{P}_{NL}$:

$$-\tfrac{1}{2}i\psi_n\mathcal{P}_n e^{i(\beta_n z-\omega_n t)}$$
$$= \frac{1}{8}\left(4\chi^{(3)}\right) D_{|k|,|l|,|m|}\psi_k\psi_l\psi_m\mathcal{E}_k\mathcal{E}_l\mathcal{E}_m e^{i\{[\beta(\omega_k)+\beta(\omega_l)+\beta(\omega_m)]z-(\omega_k+\omega_l+\omega_m)t\}}$$

$$\text{where} \qquad \omega_n = \omega_k + \omega_l + \omega_m$$

NONLINEAR POLARIZATION FOR FWM (5)

- Multiply both sides of the equation for $\mathcal{P}_n$ by $\psi_n(\mathbf{r}_T)^*$ and integrate with respect to the transverse area:

$$\boxed{\mathcal{P}_n = i\chi^{(3)} D_{|k|,|l|,|m|}\mu_{nklm}\mathcal{E}_k\mathcal{E}_l\mathcal{E}_m e^{i(\Delta\beta_{nklm})z}}$$

where

$$\omega_n = \omega_k + \omega_l + \omega_m \,,$$

the **wave-vector mismatch** is $\Delta\beta_{nklm} = -\beta(\omega_n) + \beta(\omega_k) + \beta(\omega_l) + \beta(\omega_m)$, and the **mode overlap integral** is

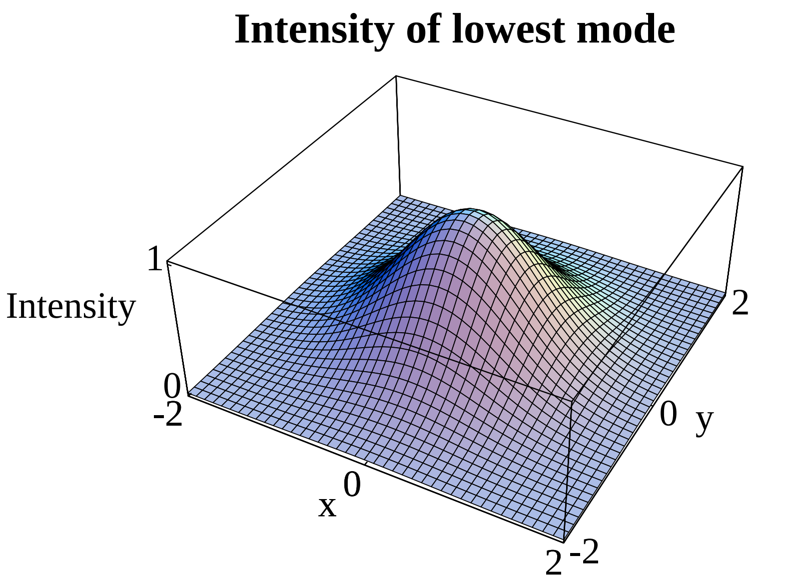
$$\mu_{nklm} = \frac{\displaystyle\iint \psi_n(\mathbf{r}_T)^*\psi_k(\mathbf{r}_T)\psi_l(\mathbf{r}_T)\psi_m(\mathbf{r}_T)\, d^2r_T}{\displaystyle\iint |\psi_n(\mathbf{r}_T)|^2\, d^2r_T}$$

- Example: $\psi_n = \psi_k = \psi_l = \psi_m = \psi_1$

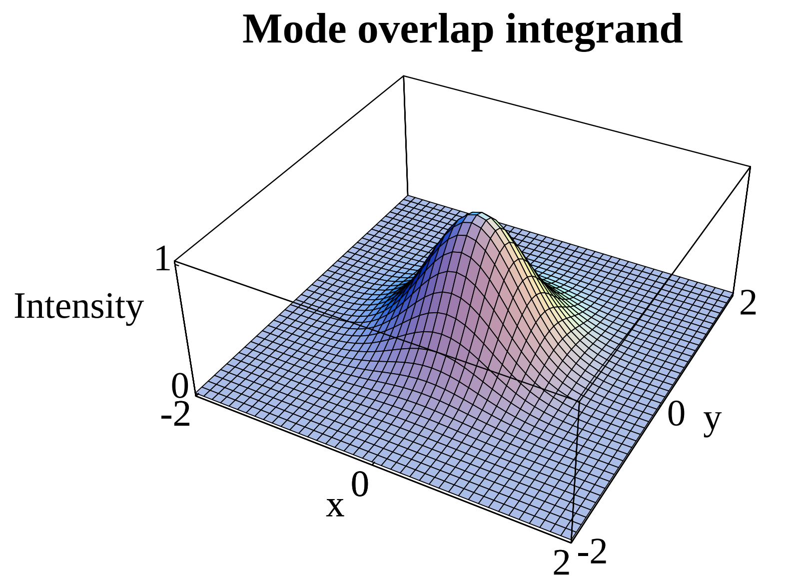
  ▷ ψ_1 is the eigenfunction for the HE_{11} mode,

$$\psi_1 \approx e^{-r^2/(2w^2)} \Rightarrow \mu_{nklm} = \frac{1}{2}$$

NONLINEAR POLARIZATION FOR FWM (6)



The intensity in the lowest mode, $|\psi_1(x,\, y)|^2$



The mode overlap integrand, $|\psi_1(x,\, y)|^4$

NONLINEAR POLARIZATION FOR FWM (7)

- Mode overlap integral:

$$\mu_{nklm} = \frac{\iint \psi_n(\mathbf{r}_T)^* \psi_k(\mathbf{r}_T) \psi_l(\mathbf{r}_T) \psi_m(\mathbf{r}_T)\, d^2 r_T}{\iint |\psi_n(\mathbf{r}_T)|^2\, d^2 r_T}$$

  ▷ Normalization of ψ_n:

$$\iint |\psi_n(\mathbf{r}_T)|^2\, d^2 r_T = 1$$

  ▷ Then $|\psi_n(\mathbf{r}_T)|^2$ has units of $(\text{area})^{-1}$

- **Effective area** for the interaction of waves k, l, m to produce wave n:

$$(A_e)^{-1} := |\mu_{nklm}|$$

SELF-PHASE MODULATION (1)

- Intensity-dependent index of refraction:
  - ▷ Nonlinear polarization envelope:

$$\begin{aligned}\mathcal{P}_n &= i\chi^{(3)} D_{|k|,|l|,|m|} \mu_{nklm} \mathcal{E}_k \mathcal{E}_l \mathcal{E}_m e^{i(\Delta\beta_{nklm})z} \\ &= 3i\chi^{(3)} \mu_{nnnn} |\mathcal{E}_n|^2 \mathcal{E}_n\end{aligned}$$

where $k = -l = m = n$ and therefore $D_{|k|,|l|,|m|} = 3$, $\Delta\beta_{nklm} = 0$

  - ▷ **Generalized nonlinear Schrödinger equation**:

$$\begin{aligned}&\left[\frac{\partial}{\partial z'} + \left(i\frac{\beta_2}{2}\frac{\partial^2}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t'^3}\right)\right]\overline{\mathcal{E}}(z', t') \\ &\qquad = -\frac{\alpha}{2}\overline{\mathcal{E}} + \frac{6\pi\chi^{(3)}\mu_{nnnn}}{\beta_0 c^2}\left(\omega_m^2 + 2i\omega_m\frac{\partial}{\partial t}\right)|\overline{\mathcal{E}}_n|^2\overline{\mathcal{E}}_n\end{aligned}$$

INTENSITY AND POWER IN FIBERS

- Electric field:

$$\mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{e}}\psi_m(\mathbf{r}_T)\mathcal{E}_m(z,\, t)e^{i(\beta z - \omega t)}\right]$$
$$= \tfrac{1}{2}\left[\hat{\mathbf{e}}\psi_m\mathcal{E}_m e^{i(\beta z-\omega t)} + \text{c.c.}\right]$$

- Intensity at $\mathbf{r}_T = (x,\, y)$:

$$I_m(\mathbf{r}_T) = \frac{c n_{0,m}}{8\pi}|\mathcal{E}_m|^2|\psi_m(\mathbf{r}_T)|^2$$

- Power in mode m:

$$P_m = \iint I_m(\mathbf{r}_T)\, d^2 r_T$$
$$= \frac{c n_{0,m}}{8\pi}|\mathcal{E}_m|^2 \iint |\psi_m(\mathbf{r}_T)|^2\, d^2 r_T$$

PROPAGATION EQUATION FOR FWM (2)

- Paraxial wave equation for the complex amplitude of a single mode (or WDM channel):

$$\frac{\partial}{\partial z'}\overline{\mathcal{E}}_n = -\frac{\alpha}{2}\overline{\mathcal{E}}_n + \frac{2\pi i\omega_n^2\chi^{(3)}}{\beta_n c^2}\mu_{nklm}D_{|k|,|l|,|m|}\overline{\mathcal{E}}_k\overline{\mathcal{E}}_l\overline{\mathcal{E}}_m e^{i(\Delta\beta_{nklm})z'}$$

- Introduce mode amplitudes which are directly related to power:

$$\mathcal{F}_n = \sqrt{\frac{cn_{0,n}A_e}{8\pi}}\mathcal{E}_n \;\Rightarrow\; \text{power in mode } n \text{ is } P_n = |\mathcal{F}_n|^2$$

- Paraxial wave equation in terms of $\mathcal{F}$'s:

$$\frac{\partial}{\partial z'}\overline{\mathcal{F}}_n = -\frac{\alpha}{2}\overline{\mathcal{F}}_n + i\frac{16\pi^2\omega_n\chi^{(3)}}{n_{0,n}^2c^2A_e}\mu_{nklm}D_{|k|,|l|,|m|}\overline{\mathcal{F}}_k\overline{\mathcal{F}}_l\overline{\mathcal{F}}_m e^{i(\Delta\beta_{nklm})z'}$$

where we have used

$$\omega_n \approx \frac{c\beta_n}{n_{0,n}}$$

SELF-PHASE MODULATION (2)

- Generalized nonlinear Schrödinger equation for normalized fields:

$$\left[\frac{\partial}{\partial z'} + \left(i\frac{\beta_2}{2}\frac{\partial^2}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t'^3}\right)\right]\overline{\mathcal{F}}(z', t')$$
$$= -\frac{\alpha}{2}\overline{\mathcal{F}}_n + i\gamma\left(1 + i\frac{2}{\omega_n}\frac{\partial}{\partial t}\right)|\overline{\mathcal{F}}|^2\overline{\mathcal{F}}$$

where

$$\gamma = \frac{48\pi^2\chi^{(3)}\,\omega_n^2}{\beta(\omega_n)c^3 A_{\text{eff}}}$$

▷ $i\gamma|\overline{\mathcal{F}}|^2\overline{\mathcal{F}}$: Self-phase modulation

▷ $-(2\gamma/\omega_n)\partial(|\overline{\mathcal{F}}|^2\overline{\mathcal{F}})/\partial t$: Self-steepening

SELF-PHASE MODULATION (3)

- Simplified paraxial wave equation for studying self-phase modulation:

$$\frac{\partial \overline{\mathcal{F}}(z', t')}{\partial z'} = -\frac{\alpha}{2}\overline{\mathcal{F}} + i\gamma|\overline{\mathcal{F}}|^2\overline{\mathcal{F}}$$

▷ Remove attenuation term:

$$\overline{\mathcal{F}}' = e^{\alpha z'/2}\overline{\mathcal{F}}$$

▷ Equation for $\mathcal{F}'$:

$$\frac{\partial \overline{\mathcal{F}}'}{\partial z'} = i\gamma e^{-\alpha z'}|\overline{\mathcal{F}}'|^2\overline{\mathcal{F}}'$$

▷ Equation for intensity:

$$\frac{\partial}{\partial z'}|\overline{\mathcal{F}}'|^2 = \frac{\partial}{\partial z'}\overline{\mathcal{F}}^{*\prime}\overline{\mathcal{F}}' = \frac{\partial \overline{\mathcal{F}}^{*\prime}}{\partial z'}\overline{\mathcal{F}}' + \frac{\partial \overline{\mathcal{F}}'}{\partial z'}\overline{\mathcal{F}}^{*\prime} = 0$$

SELF-PHASE MODULATION (4)

- Equation for $\overline{\mathcal{F}}' = e^{\alpha z'/2}\overline{\mathcal{F}}$:

$$\frac{\partial \overline{\mathcal{F}}'}{\partial z'} = i\gamma e^{-\alpha z'} |\overline{\mathcal{F}}'|^2 \overline{\mathcal{F}}'$$

▷ The modulus of $\mathcal{F}'$ is constant:

$$|\overline{\mathcal{F}}'(z', t')|^2 = |\overline{\mathcal{F}}'(0, t')|^2 = P_0(t')$$

where P_0 is the initial (time-dependent) power

▷ Equation for $\mathcal{F}'$:

$$\frac{\partial \overline{\mathcal{F}}'(z', t')}{\partial z'} = i\gamma P_0(t') e^{-\alpha z'} \overline{\mathcal{F}}'(z', t')$$

▷ Solution:

$$\boxed{\overline{\mathcal{F}}(z', t') = e^{-\alpha z'/2} \overline{\mathcal{F}}(0, t') e^{i\gamma P_0(t') L_{\text{eff}}(z')}}$$

SELF-PHASE MODULATION (5)

- Evolution of $\overline{\mathcal{F}}$ under the influence of SPM:

$$\overline{\mathcal{F}}(z', t') = e^{-\alpha z'/2}\overline{\mathcal{F}}(0, t')e^{i\gamma P_0(t')L_{\text{eff}}(z')}$$

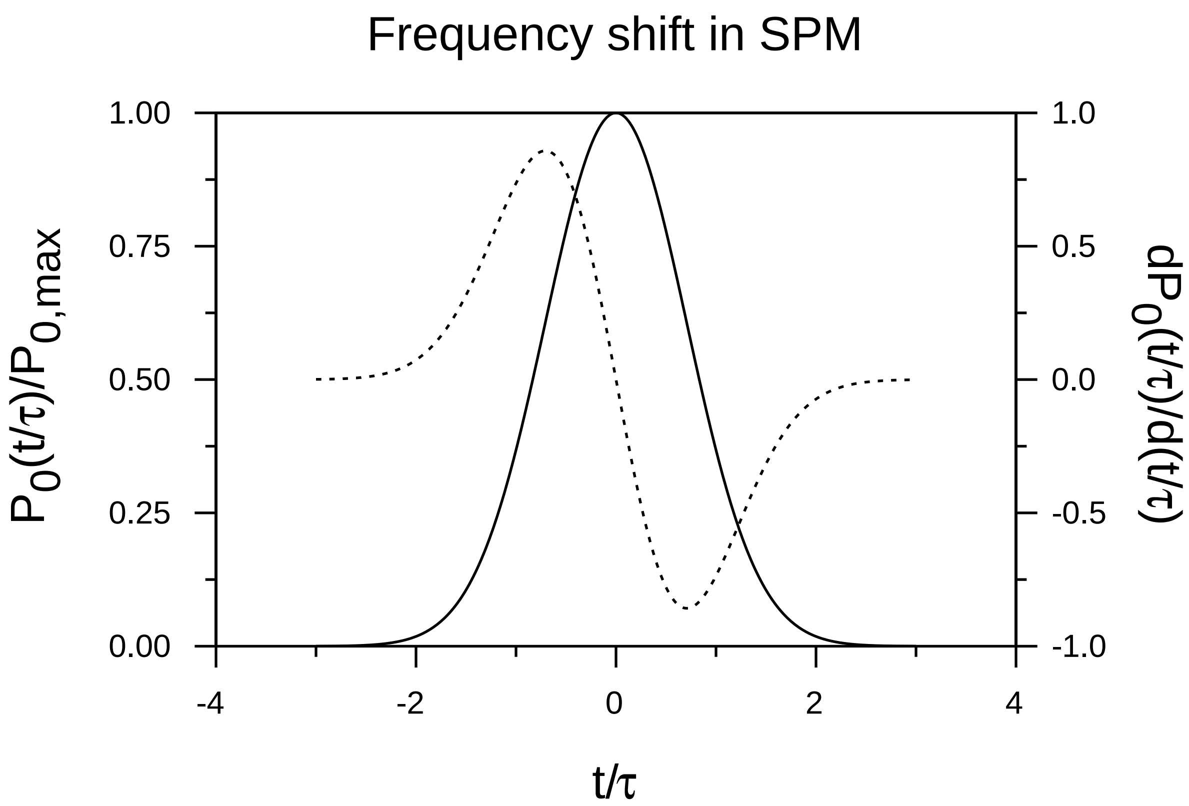
▷ Instantaneous frequency of $\overline{\mathcal{F}}$:

$$\begin{aligned}\delta\omega(z', t') &= i\frac{\partial}{\partial t'}\ln\overline{\mathcal{F}}(z', t') \\ &= -\gamma L_{\text{eff}}(z')\frac{dP_0(t')}{dt'}\end{aligned}$$

where I_0 is the initial (time-dependent) intensity

▷ (Instantaneous frequency of $\mathbf{E}$) $= \omega_n + \delta\omega(z', t')$

SELF-PHASE MODULATION (6)



SELF-PHASE MODULATION (7)

- Evolution of $\overline{\mathcal{F}}$ under the influence of SPM:

$$\overline{\mathcal{F}}(z', t') = e^{-\alpha z'/2}\overline{\mathcal{F}}(0, t')e^{i\gamma P_0(t')L_{\rm eff}(z')}$$

  ▷ **Nonlinear length**:

$$L_{NL} := (\gamma P_{0,\max})^{-1}$$

  where P_0 is the initial power

  ▷ Numerical values:

$$\gamma \approx 3 \text{ W}^{-1}\text{km}^{-1}$$

  For $P_0 = 10$ mW,

$$L_{NL} \approx 30 \text{ km}$$

SELF-PHASE MODULATION (8)

- Effects of SPM in fiberoptic telecom systems:
  - ▷ Maximum phase shift:

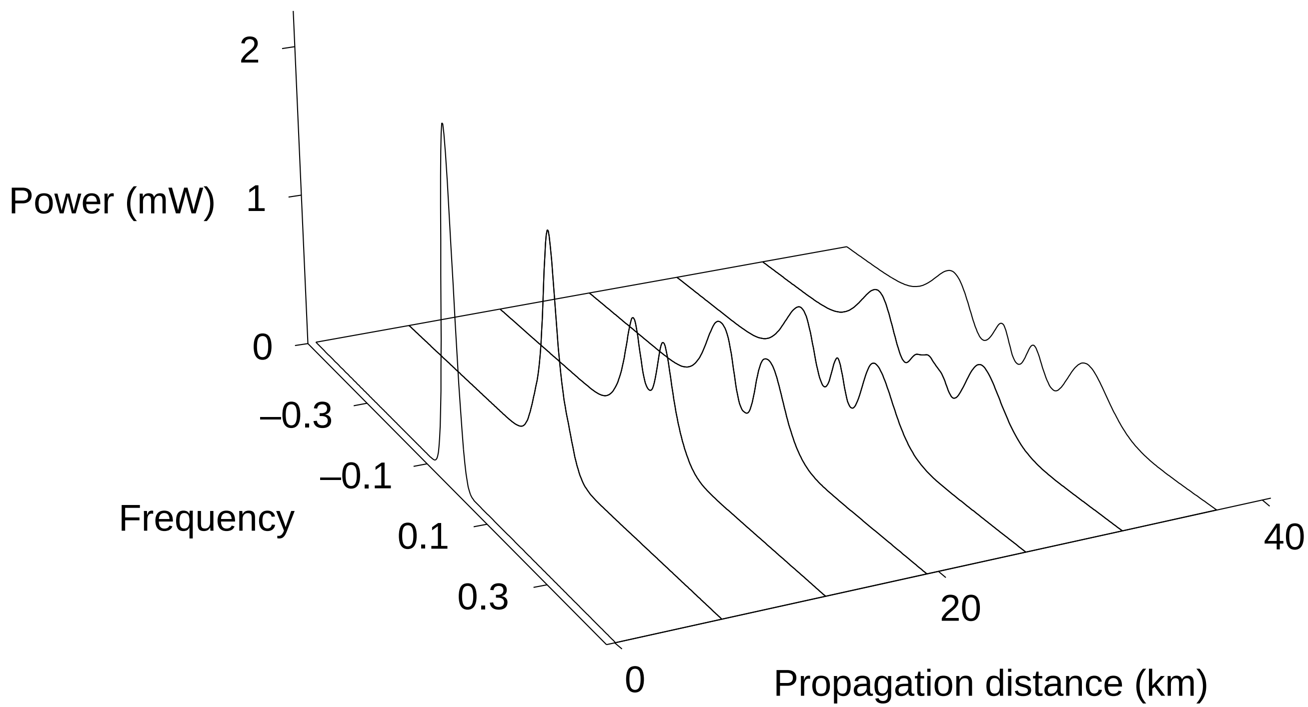
$$\Delta\phi_{\max} = \frac{1}{\alpha L_{NL}}$$

    - ∘ For $P_0 = 10$ mW, $L_{NL} \approx 30$ km
    - ∘ Maximum phase shift for $\alpha \approx (20\text{ km})^{-1}$:

$$\Delta\phi_{\max} \approx \tfrac{2}{3}\text{ radian}$$

  - ▷ Chirp $\Rightarrow$ spectral broadening
  - ▷ Compensation of anomalous dispersion (solitons)

EFFECT OF SELF-PHASE MODULATION ON THE SPECTRUM OF A PULSE



CROSS-PHASE MODULATION (1)

- Simplified paraxial wave equation for studying cross-phase modulation:

$$\frac{\partial \overline{\mathcal{F}}_1(z',t')}{\partial z'} = -\frac{\alpha}{2}\overline{\mathcal{F}}_1 + i\gamma(|\overline{\mathcal{F}}|_1^2 + 2|\overline{\mathcal{F}}|_2^2)\overline{\mathcal{F}}_1$$

  - ▷ $|\overline{\mathcal{F}}|_1^2\overline{\mathcal{F}}_1$: SPM
  - ▷ $2|\overline{\mathcal{F}}|_2^2\overline{\mathcal{F}}_1$: Cross-phase modulation (XPM)

SELF-STEEPENING (1)

- Because of nonlinear effects, the leading edge of a pulse travels more slowly than the trailing edge
  - ▷ Optical shock formation:
    The development of an almost-instantaneous transition from near-zero amplitude to a finite amplitude
  - ▷ Critical distance for optical shock formation:

$$L_{\text{shock}} \sim L_{NL} \times (\text{no. of cycles in pulse})$$

SELF-STEEPENING (2)

- Generalized nonlinear Schrödinger equation for normalized fields:

$$\left[\frac{\partial}{\partial z'} + \left(i\frac{\beta_2}{2}\frac{\partial^2}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t'^3}\right)\right]\overline{\mathcal{F}}(z', t')$$
$$= -\frac{\alpha}{2}\overline{\mathcal{F}} + i\gamma\left(1 + i\frac{2}{\omega_0}\frac{\partial}{\partial t'}\right)|\overline{\mathcal{F}}|^2\overline{\mathcal{F}}$$

▷ Let T_0 = initial pulse width,

$$\tau = \frac{t'}{T_0},$$

$$\zeta := \frac{z'}{L_{NL}} = \gamma P_0 z',$$

$$e^{-\alpha z'/2}\sqrt{P_0}\,U(\zeta, \tau) := \overline{\mathcal{F}}(z', t'),$$

$$s := \frac{2}{\omega_0 T_0}$$

SELF-STEEPENING (3)

- Equation describing optical shock formation:

$$\frac{\partial U}{\partial \zeta} + s\frac{\partial}{\partial \tau}(|U|^2 U) = i|U|^2 U$$

▷ Introduce phase ϕ and normalized intensity I:

$$U = \sqrt{I}\, e^{i\phi}$$

▷ New equations:

$$\frac{\partial I}{\partial \zeta} + 3sI\frac{\partial I}{\partial \tau} = 0$$

$$\frac{\partial \phi}{\partial \zeta} + sI\frac{\partial \phi}{\partial \tau} = I$$

▷ Intensity:

$$I(\zeta, \tau) = f(\tau - 3sI\zeta)$$