OVERVIEW OF SBS IN FIBERS

- Cause:
  - ▷ Nonlinear interaction of light with hypersonic acoustic waves
- Effects:
  - ▷ Depletes signal
  - ▷ Increases BER
  - ▷ Sends power back into transmitter
- Impacts on amplified telecom systems:
  - ▷ Strongest optical nonlinearity
  - ▷ Restricts launch power
  - ▷ Decreases transmission distance without regeneration
  - ▷ Forces adoption of gain-reducing techniques

QUALITATIVE PICTURE OF BRILLOUIN SCATTERING (1)

- Oppositely-propagating light waves superpose to form a moving system of interference fringes:

$$\underbrace{\cos(\beta_L z - \omega_L t)}_{\text{travels forward}} + \underbrace{\cos(\beta_B z + \omega_B t)}_{\text{travels backward}} = 2\underbrace{\cos\tfrac{1}{2}\left[(\beta_L + \beta_B)\,z - (\omega_L - \omega_B)\,t\right]}_{\text{travels forward if } \omega_L > \omega_B}$$
$$\cdot\, 2\cos\tfrac{1}{2}\left[(\beta_L - \beta_B)\,z - (\omega_L + \omega_B)\,t\right]$$

- Velocities:
  - ▷ Light waves:

$$v_L = \frac{c}{n_L} = \frac{\omega_L}{\beta_L}, \quad v_B = \frac{c}{n_B} = \frac{\omega_B}{\beta_B}$$

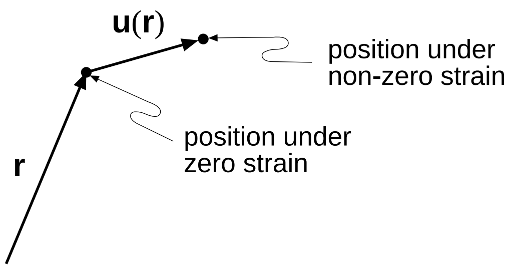
  - ▷ The interference fringe pattern defined by $\cos\frac{1}{2}\left[(\beta_L + \beta_B)\,z - (\omega_L - \omega_B)\,t\right]$:

$$v_i = \frac{\omega_L - \omega_B}{\beta_L + \beta_B} \ll v_L \text{ or } v_B$$

QUALITATIVE PICTURE OF BRILLOUIN SCATTERING (2)

- If
$$v_i = \text{longitudinal acoustic velocity} = c_s$$
then the interference pattern travels forward at the same velocity as a sound wave:
  - ▷ Frequency $\omega_s = c_s k_s = \omega_L - \omega_B$
  - ▷ Wave vector $k_s = \beta_L + \beta_B$
- Electrostriction couples the light waves with the acoustic wave
  - ▷ Matter is pushed out of regions of strong electric field, driving the acoustic wave
  - ▷ The acoustic wave changes the dielectric permittivity, driving the electric field

DISPLACEMENT FIELD

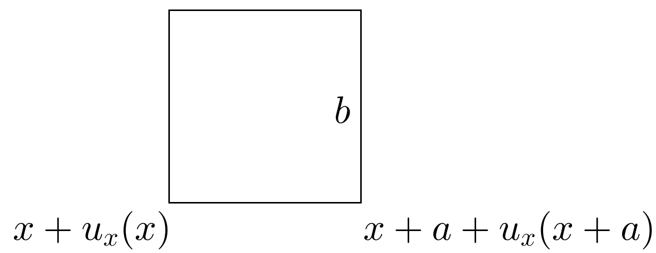


- The displacement of a particle which was originally at $\mathbf{r}$ is $\mathbf{u}(\mathbf{r})$
- Relation of the displacement field $\mathbf{u}$ to the volume strain $\Delta v/v$:

$$\frac{\Delta v}{v} = \nabla \cdot \mathbf{u}$$

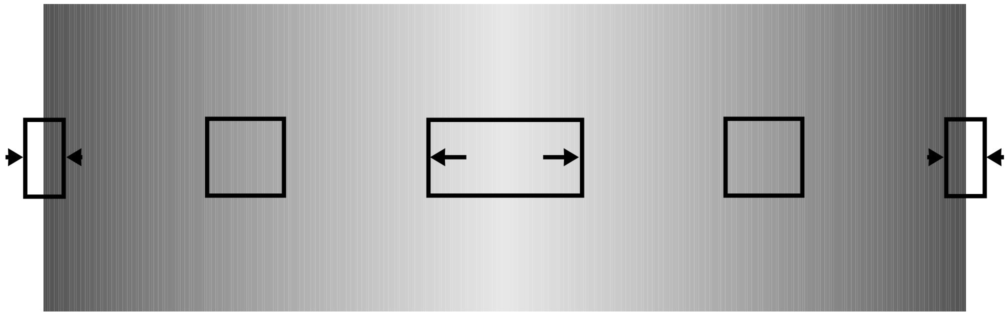
VOLUME STRAIN

- Volume strain (fractional change in volume) of a small box with undeformed dimensions (a, b, c):



$$\frac{\Delta v}{v} = \frac{\{(x+a) + u_x(x+a) - [x + u_x(x)]\}bc - abc}{abc}$$
$$= \frac{u_x(x+a) - u_x(x)}{a} = \frac{du_x}{dx} = \nabla \cdot \mathbf{u}$$

ONE-DIMENSIONAL ACOUSTIC WAVE



- Conventions:
  - ▷ Shading indicates a pressure (compression-rarefaction) wave
  - ▷ Arrows show the displacement field $\mathbf{u}$
  - ▷ Boxes show the deformation of a small volume

ELECTRICAL ENERGY STORED IN A CAPACITOR

- Stored energy:

$$W = \tfrac{1}{2} C V^2 = \frac{Q^2}{2C}$$

  - ▷ C = capacitance = ϵC_0, where C_0 = capacitance in vacuum

- Change in W due to change in ϵ:

  - ▷ Under constant voltage $V = \int \mathbf{E} \cdot d\mathbf{l}$:

$$\Delta W_V = \tfrac{1}{2} C_0 (\Delta\epsilon) V^2 = \frac{\Delta\epsilon}{\epsilon} W$$

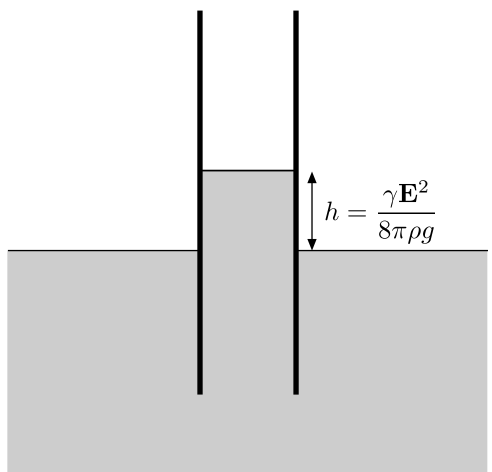
  - ▷ Under constant charge:

$$\Delta W_Q = -\frac{Q^2}{\epsilon C_0} \Delta\epsilon = -\frac{\Delta\epsilon}{\epsilon} W$$

BEHAVIOR OF A LIQUID DIELECTRIC IN A CAPACITOR

- Assume constant voltage $V = \int \mathbf{E} \cdot d\mathbf{l}$
  - ▷ Dielectric level rises between capacitor plates by a distance

$$h = \frac{\gamma \mathbf{E}^2}{8\pi\rho g}$$



ELECTROSTRICTIVE CONSTANT

- Definition:

$$\gamma = \rho \frac{d\epsilon}{d\rho}$$

where $\rho =$mass density and $\epsilon =$dielectric constant

- Clausius-Mossotti relation:

$$\frac{4\pi\alpha}{3m}\rho = \frac{\epsilon - 1}{\epsilon + 2}$$

where $m =$molecular mass and $\alpha =$ polarizability

- Differentiate:

$$\frac{4\pi\alpha}{3m} = \frac{d\epsilon}{d\rho}\frac{3}{(\epsilon + 2)^2}$$

- Substitute from Clausius-Mossotti relation:

$$\rho\frac{d\epsilon}{d\rho} = \gamma = \tfrac{1}{3}(\epsilon - 1)(\epsilon + 2)$$

ELECTROSTRICTIVE CHANGE IN DIPOLE MOMENT / UNIT VOLUME

- Hooke's law (strain is proportional to stress):

$$p = -T_B \nabla \cdot \mathbf{u}$$
$$T_B = \text{bulk modulus}$$

- Strain changes the dielectric susceptibility:

$$\Delta\epsilon = \frac{d\epsilon}{d\rho}\Delta\rho = -\gamma \nabla \cdot \mathbf{u} = 4\pi\Delta\chi$$
$$\Rightarrow \Delta\chi = -\frac{\gamma}{4\pi}\nabla \cdot \mathbf{u} = \frac{\gamma}{4\pi T_B}p = \frac{\gamma}{4\pi\overline{\rho}}\Delta\rho$$

- Change in polarization (dipole moment per unit volume) due to volume strain:

$$\Delta\mathbf{P} = \mathbf{E}\Delta\chi = \frac{\gamma}{4\pi T_B}p\mathbf{E} = \frac{\gamma}{4\pi\overline{\rho}}\mathbf{E}\Delta\rho$$

BODY FORCES IN DIELECTRICS

- Assume constant "charges" (sources of $\mathbf{E}$); then

$$\Delta U_{\mathrm{el}} = -\frac{1}{8\pi}\int (\Delta\epsilon)\,\mathbf{E}^2\,d^3r$$

- Strain $\mathbf{u}$ induces $\Delta\epsilon$:

$$\Delta\epsilon = \frac{d\epsilon}{d\rho}\Delta\rho = \rho\frac{d\epsilon}{d\rho}\frac{\Delta\rho}{\rho} = -\rho\frac{d\epsilon}{d\rho}\frac{\Delta v}{v} = -\rho\frac{d\epsilon}{d\rho}\nabla\cdot\mathbf{u} = -\gamma\nabla\cdot\mathbf{u}$$

- $\Delta\epsilon$ changes the electric energy density:

$$\begin{aligned}\Delta U_{\mathrm{el}} &= -\frac{1}{8\pi}\int (\Delta\epsilon)\,\mathbf{E}^2\,d^3r = \frac{\gamma}{8\pi}\int \mathbf{E}^2\,\nabla\cdot\mathbf{u}\,d^3r \\ &= \frac{\gamma}{8\pi}\int \nabla\cdot(\mathbf{E}^2\mathbf{u})\,d^3r - \frac{\gamma}{8\pi}\int \mathbf{u}\cdot\nabla(\mathbf{E}^2)\,d^3r \\ &= \int \mathbf{f}_{\mathrm{el}}\cdot\mathbf{u}\,d^3r \quad (\mathbf{f}_{\mathrm{el}} = \text{electric body force})\end{aligned}$$

- Assume that $\mathbf{E}^2\mathbf{u} = 0$ on the boundary; then

$$\mathbf{f}_{\mathrm{el}} = -\frac{\gamma}{8\pi}\nabla(\mathbf{E}^2) = -\nabla p_{\mathrm{el}}$$

- Pressure change due to $\mathbf{E}^2$:

$$p_{\mathrm{el}} = \frac{\gamma}{8\pi}\mathbf{E}^2$$

ELECTROSTRICTION AND SBS

- Nonlinear mixing of frequencies occurs in electrostriction:
  - ▷ If $\mathbf{E} = \mathbf{E}_L + \mathbf{E}_B$ then $\mathbf{E}^2$ has frequency components at $\pm\omega_s$, where
$$\omega_s = \omega_L - \omega_B$$
  - ▷ Then
$$p_{\text{el}} = \frac{\gamma}{8\pi}\mathbf{E}^2$$
    oscillates at $\pm\omega_s$ (an acoustic wave!)
- Feedback:
$$\Delta\mathbf{P} = \frac{\gamma}{4\pi T_B} p_{\text{el}} \mathbf{E}$$
  radiates electromagnetic waves at frequencies
  $\omega_L - \omega_s = \omega_B$ and $\omega_B + \omega_s = \omega_L$
- If $\Delta\chi = \Delta\chi' + i\Delta\chi''$ and $p = p' + ip''$, then
$$\Delta\chi' = \frac{\gamma}{4\pi T_B} p' \Rightarrow \text{frequency shift}$$
$$\Delta\chi'' = \frac{\gamma}{4\pi T_B} p'' \Rightarrow \text{gain (if } \Delta\chi'' < 0)$$

FREQUENCY AND WAVE VECTOR CONTENT OF THE ELECTRIC POLARIZATION IN SBS

- From

$$\Delta \mathbf{P} = \frac{\gamma}{4\pi T_B} p_{\text{el}} \mathbf{E}$$

we see that if

$$p_{\text{el}} = \text{Re}\left[\wp e^{i\mathbf{k}\cdot\mathbf{r}-\omega t}\right]$$

then the frequency content of $\Delta \mathbf{P}$ is:

wave vector	frequency	identification
$\mathbf{k}_L - \mathbf{k} = \mathbf{k}_B$	$\omega_L - \omega = \omega_B$	Stokes
$\mathbf{k}_B + \mathbf{k} = \mathbf{k}_L$	$\omega_B + \omega = \omega_L$	pump
$\mathbf{k}_L + \mathbf{k}$	$\omega_L + \omega$	1^{st} anti-Stokes
$\mathbf{k}_B - \mathbf{k}$	$\omega_B - \omega$	2^{nd} Stokes

- Confirms that Brillouin scattering is the scattering of a light wave by a traveling acoustic wave

SPONTANEOUS vs. STIMULATED BRILLOUIN SCATTERING

- In *spontaneous* Brillouin scattering, $p_{\rm el}$ is created thermally:

$$\Delta\mathbf{P}_{\rm spont} = \mathrm{Re}\left[(\mathrm{const.})e^{i\mathbf{k}\cdot\mathbf{r}-\omega t}\right]\mathbf{E}_L$$

- In *stimulated* Brillouin scattering, $p_{\rm el}$ is created by a pump wave:

$$[\Delta\mathbf{P}_{\rm stim}]_{\mathbf{k}_B,\omega_B} = (\mathrm{const.})[\mathbf{E}_L\cdot\mathbf{E}_B]_{\mathbf{k},\omega}\,\mathbf{E}_B$$

  - ▷ The Stokes field (at $\omega_B = \omega_L - \omega$) grows at the expense of the pump field (at ω_L) until the pump is depleted
  - ▷ The second Stokes field (at $\omega_B - \omega$) remains small until the pump is depleted by scattering into the Stokes field
  - ▷ The first anti-Stokes field (at $\omega_L + \omega$) remains small if no anti-Stokes signal is injected, unless all waves are amplified (as in an EDFA)

SBS: ACOUSTIC EQUATIONS (1)

- Assume that $\mathbf{E}^2$ changes slowly over the decay time of density fluctuations
- Then the pressure fluctuates around the electrostrictive pressure:

$$\Delta p = p - \frac{\gamma}{8\pi}\mathbf{E}^2 = -T_B \nabla \cdot \mathbf{u} - \frac{\gamma}{8\pi}\mathbf{E}^2 = T_B \frac{\Delta\rho}{\overline{\rho}} - \frac{\gamma}{8\pi}\mathbf{E}^2 = c_s^2 \Delta\rho - \frac{\gamma}{8\pi}\mathbf{E}^2$$

where c_s = speed of sound

- Recall the equation

$$-\Gamma m \frac{dx}{dt} - (m\omega_0^2 x - F) = m\frac{d^2 x}{dt^2}$$

for damped, forced harmonic motion of a mass m

- ▷ Continuum case: assume a damping force density of $-\Gamma_B \partial\mathbf{u}/\partial t$
- ▷ Pressure force density $= -\nabla(\Delta p)$
- ▷ Resulting continuum equation:

$$-\Gamma_B \overline{\rho}\frac{\partial \mathbf{u}}{\partial t} - \nabla(\Delta p) = \overline{\rho}\frac{\partial^2 \mathbf{u}}{\partial t^2}$$

SPONTANEOUS BRILLOUIN LINEWIDTH IN BULK MEDIA

- General formula showing dependence on acoustic wave vector:

$$\Gamma_B = \frac{\eta k^2}{\overline{\rho}}$$

$$k = \frac{2\pi}{\lambda_{\text{acoustic}}}$$

$$\overline{\rho} = \text{density}$$

$$\eta = \text{constant}$$

- Value in bulk silica at $\lambda_{L,\text{vac}} = 1550$ nm:

$$\boxed{\Gamma_B = 16\,\text{MHz}}$$

SPONTANEOUS BRILLOUIN LINEWIDTH IN DISPERSION-SHIFTED FIBER

- Different radial regions have different longitudinal acoustic wave velocities
  - ▷ Different regions have different values of the Brillouin acoustic frequency

$$\omega_s \approx 2 n_0 c_s k_{L,\mathrm{vac}}$$

- Representative shifts of ω_s:
  - ▷ GeO_2 doping: $\Delta\omega_s = 125$ MHz/mol%
  - ▷ P_2O_5 doping: $\Delta\omega_s = 162$ MHz/mol%
- Result:

$$\Gamma_B \approx 100\,\mathrm{MHz}$$

in DSF at $\lambda_{L,\mathrm{vac}} = 1550$ nm

GUIDED ACOUSTIC WAVES IN SBS

- If

$$c_s^{\text{core}} < c_s^{\text{clad}}$$

  then guided acoustic waves can exist in the core

  ▷ Example: Pure silica cladding, germania-doped core

- Each guided acoustic mode produces a peak in the spontaneous Brillouin gain spectrum
- Only the strongest peak gives rise to SBS
- Reference: N. Shibata et al., *Optics Letters* **13**, 595-597 (1988)

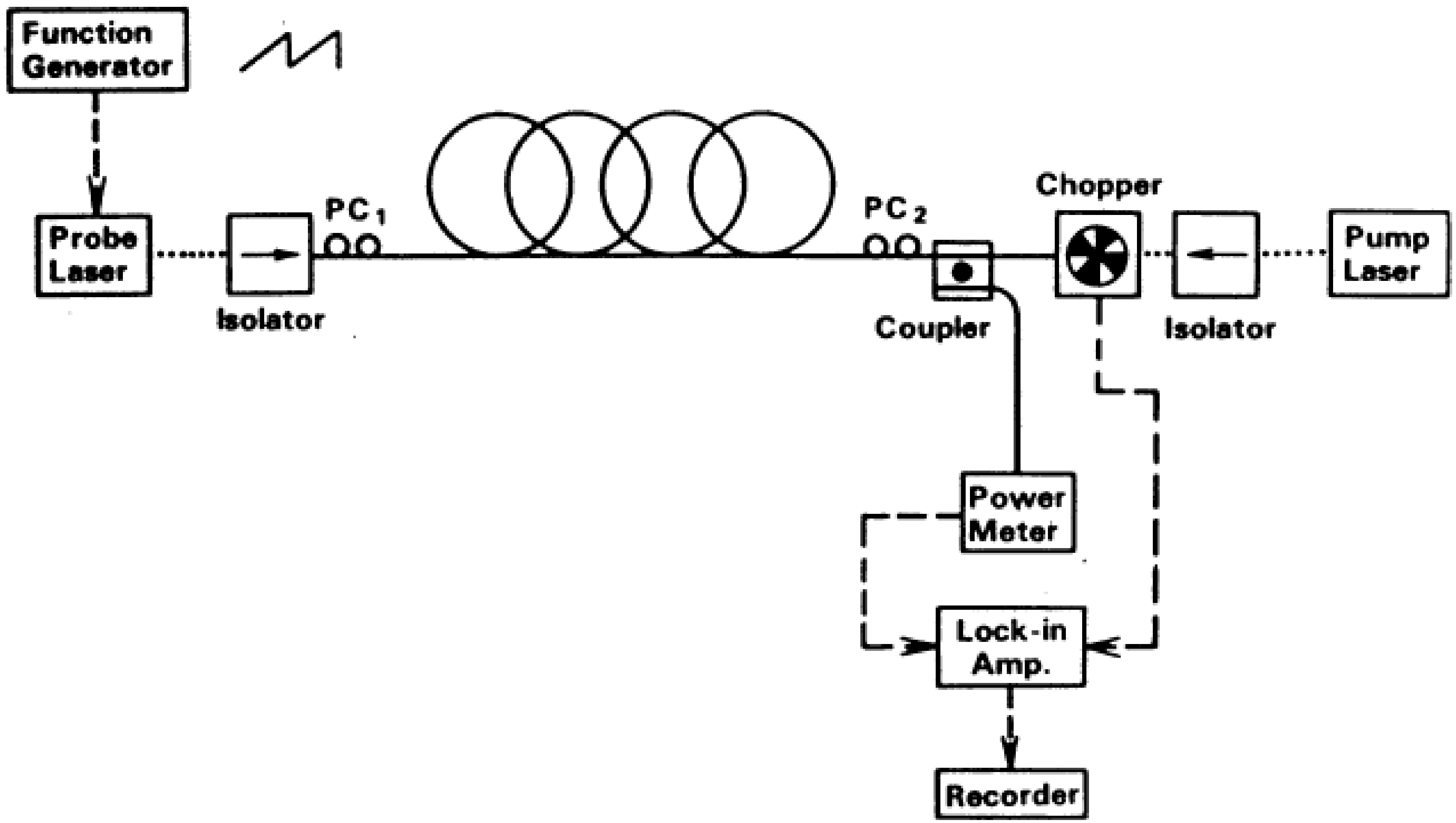


Fig. 1. Experimental arrangement for measuring Brillouin gain spectrum. PC_1 and PC_2, polarization controllers.

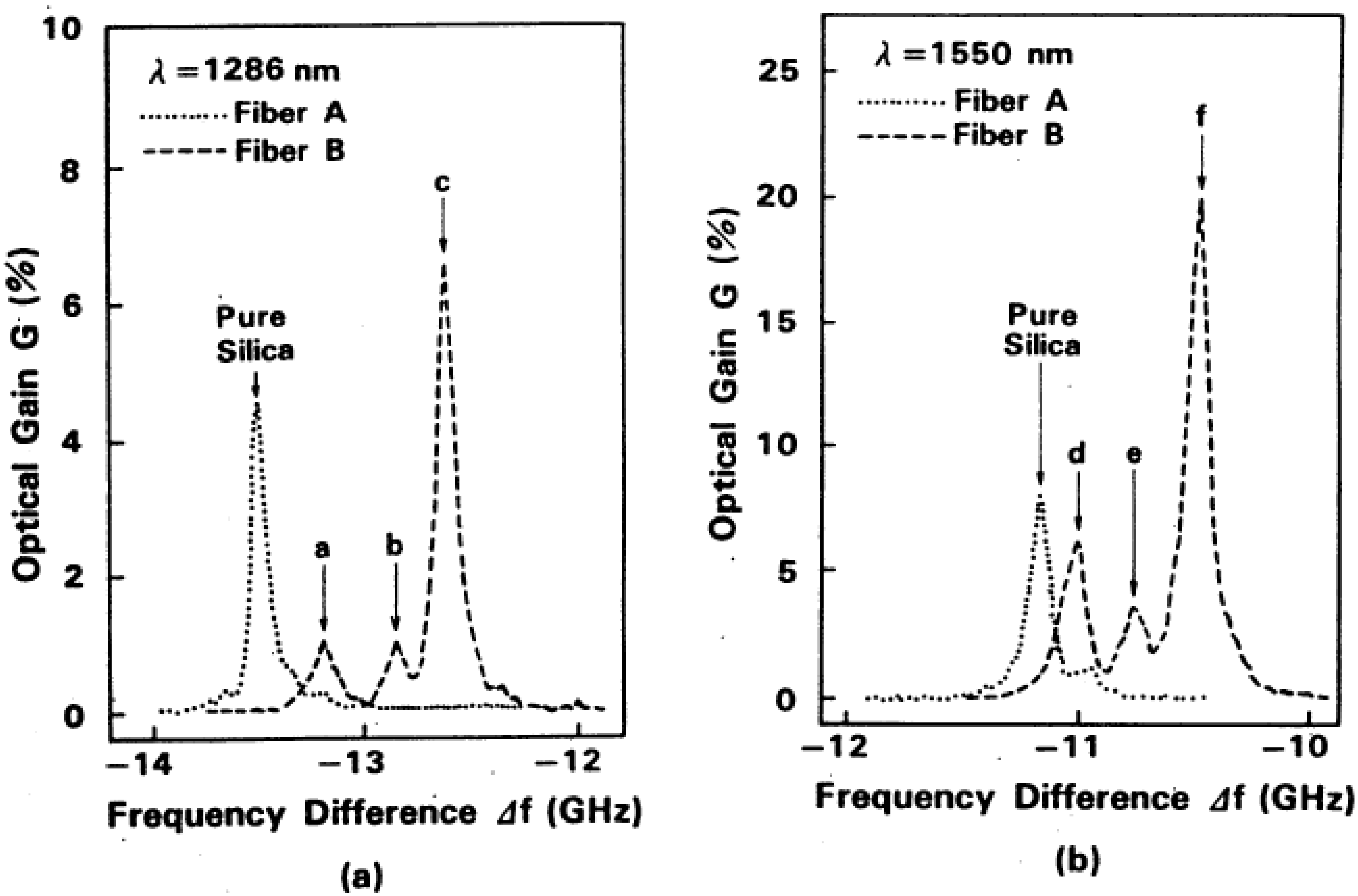


Fig. 2. Brillouin gain spectra obtained for test fibers A and B at the wavelengths of (a) 1286 nm and (b) 1550 nm.

SBS: ACOUSTIC EQUATIONS (2)

- Equation of motion for density fluctuations:

$$-\Gamma_B\overline{\rho}\frac{\partial \mathbf{u}}{\partial t} - c_s^2\nabla(\Delta\rho) + \frac{\gamma}{8\pi}\nabla(\mathbf{E}^2) = \overline{\rho}\frac{\partial^2 \mathbf{u}}{\partial t^2}$$

where Γ_B is the Brillouin linewidth (FWHM) and $\overline{\rho}$ is the equilibrium density.

- Eliminate the displacement $\mathbf{u}$ by taking the divergence and using

$$\frac{\Delta\rho}{\overline{\rho}} = -\nabla\cdot\mathbf{u}$$

to get

$$\Gamma_B\frac{\partial}{\partial t}\Delta\rho - c_s^2\nabla^2(\Delta\rho) + \frac{\gamma}{8\pi}\nabla^2(\mathbf{E}^2) = -\frac{\partial^2}{\partial t^2}\Delta\rho$$

- Wave equation for $\Delta\rho$:

$$\boxed{\left(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \frac{\Gamma_B}{c_s^2}\frac{\partial}{\partial t}\right)\Delta\rho = \frac{\gamma}{8\pi c_s^2}\nabla^2(\mathbf{E}^2)}$$

ELECTROMAGNETIC EQUATIONS FOR SBS IN FIBERS (1)

- Introduce slowly varying amplitudes:

$$\mathbf{E}_L = \hat{\mathbf{x}}\,\mathrm{Re}\,[\mathcal{E}_L e^{i\beta_L z-\omega_L t}], \quad \mathbf{E}_B = \hat{\mathbf{x}}\,\mathrm{Re}\,[\mathcal{E}_B e^{ik_B z-\omega_B t}]$$

$$\mathbf{P}_{NL} = \hat{\mathbf{x}}\,\mathrm{Re}\,[\mathcal{P}_{NL,B} e^{ik_B z-\omega_B t}], \quad \Delta\rho = \mathrm{Re}\,[q_k(t)e^{ikz}] = \mathrm{Re}\,[q_{k,0}e^{ikz-\omega t}]$$

where $\mathrm{Re}\,[z] = \frac{1}{2}(z+z^*)$

- In a fiber there can be only two directions for $\mathbf{k}_B$:
  - ▷ $\mathbf{k}_B$ parallel to $\beta_L\hat{\mathbf{z}}$:

$$k = \beta_L - \beta_B \approx 0$$

  - ▷ $\mathbf{k}_B$ antiparallel to $\beta_L\hat{\mathbf{z}}$:

$$\mathbf{k}_B = -\beta_B\hat{\mathbf{z}} \approx -\mathbf{k}_L$$

$$k = \beta_L + \beta_B \approx 2\beta_L \approx 2n_0\frac{2\pi}{\lambda_{L,\mathrm{vac}}}$$

  - ▷ Wavelength of acoustic wave when $\mathbf{k}_B$ is antiparallel to $\beta_L\hat{\mathbf{z}}$:

$$\lambda \approx \frac{\lambda_{L,\mathrm{vac}}}{2n_0} \approx 500\,\mathrm{nm} \text{ if } \lambda_{L,\mathrm{vac}} \approx 1550\,\mathrm{nm}$$

ACOUSTIC AMPLITUDE IN SBS (1)

- Wave equation for the density fluctuation $\Delta\rho$:

$$\left(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \frac{\Gamma_B}{c_s^2}\frac{\partial}{\partial t}\right)\Delta\rho = \frac{\gamma}{8\pi c_s^2}\nabla^2(\mathbf{E}^2)$$

- Forcing term:

$$\nabla^2(\mathbf{E}^2) = \nabla^2\left(2\mathbf{E}_L \cdot \mathbf{E}_B + \cdots\right) = -\tfrac{1}{2}k^2\mathcal{E}_L\mathcal{E}_B^* e^{i(kz-\omega t)} + \cdots$$

- Substitute $\Delta\rho = \mathrm{Re}\left[q_k(t)e^{ikz}\right]$ into the wave equation for $\Delta\rho$ to get the damped, forced oscillator equation

$$\left(\frac{d^2}{dt^2} + \Gamma_B\frac{d}{dt} + c_s^2k^2\right)q_k = \frac{\gamma k^2}{8\pi}\mathcal{E}_L\mathcal{E}_B^* e^{-i\omega t}$$

- Steady-state response when $q_k = q_{k,0}e^{-i\omega t}$:

$$q_{k,0} = \left(\frac{1}{c_s^2k^2 - \omega^2 - i\Gamma_B\omega}\right)\frac{\gamma k^2}{8\pi}\mathcal{E}_L\mathcal{E}_B^*$$

$$\approx \left(\frac{\gamma k}{16\pi c_s}\right)\frac{(c_sk-\omega)+i(\Gamma_B/2)}{(c_sk-\omega)^2+(\Gamma_B/2)^2}\,\mathcal{E}_L\mathcal{E}_B^*$$

ELECTROMAGNETIC EQUATIONS FOR SBS IN FIBERS (2)

- Nonlinear electric polarization
  - ▷ Electrostrictive contribution:

$$\mathbf{P}_{NL} = \frac{\gamma}{4\pi\overline{\rho}}\mathbf{E}\Delta\rho$$

  - ▷ In terms of slowly varying amplitudes:

$$\mathbf{E}\Delta\rho = \tfrac{1}{4}\hat{\mathbf{x}}\left(\mathcal{E}_L e^{i(\beta_L z-\omega_L t)} q_{k,0}^* e^{-i(kz-\omega t)} + \cdots\right)$$

$$\mathbf{P}_{NL} = \tfrac{1}{2}\hat{\mathbf{x}}\left(-i\mathcal{P}_{NL,B} e^{i(k_B z-\omega_B t)} + \text{c.c.}\right) \quad \text{where} \quad k_B = \beta_L - k$$

  - ▷ Nonlinear electric polarization at k_B and ω_B:

$$\overline{\mathcal{P}}_{NL,B} = \frac{i\gamma}{8\pi\overline{\rho}}\overline{\mathcal{E}}_L q_{k,0}^*$$

- Propagation equation for $\overline{\mathcal{E}}_B$:

$$\pm\frac{\partial}{\partial z'}\overline{\mathcal{E}}_B = -\frac{\alpha_B}{2}\overline{\mathcal{E}}_B + \frac{i\gamma\omega_B^2}{4|k_B|c^2\overline{\rho}}\overline{\mathcal{E}}_L q_{k,0}^*$$

$+$ sign for forward propagation, $-$ sign for backward propagation

ACOUSTIC AMPLITUDE IN SBS (2)

- The nonlinear electric polarization is largest for the largest acoustic amplitude $|q_{k,0}| \propto |k|$.
  - ▷ The maximum $|k| = |\beta_L - k_B|$ occurs when $k_B = -\beta_B$ (with $\beta_B > 0$). Then the Brillouin wave propagates antiparallel to the laser (pump) wave.
  - ▷ Acoustic wave vector:

$$\boxed{k = \beta_L + \beta_B \approx 2n_0 k_{L,\text{vac}}}$$

where

$$k_{L,\text{vac}} = \frac{2\pi}{\lambda_{L,\text{vac}}}$$

- For a given k, the maximum $|q_{k,0}|$ occurs when the resonant denominator $(c_s - \omega)^2 + (\Gamma_B)^2$ is as small as possible: $\boxed{\omega = c_s k}$
- Peak acoustic amplitude:

$$q_{k,0} = \frac{i\gamma k}{8\pi c_s \Gamma_B} \overline{\mathcal{E}}_L \overline{\mathcal{E}}_B^* \approx \frac{i\gamma n_0 k_{L,\text{vac}}}{4\pi c_s \Gamma_B} \overline{\mathcal{E}}_L \overline{\mathcal{E}}_B^*$$

ELECTROMAGNETIC EQUATIONSFOR SBS IN FIBERS (3)

- Nonlinear electric polarization
  - ▷ Pump intensity: $$I_L = \frac{cn_0}{8\pi}|\overline{\mathcal{E}}_L|^2$$
  - ▷ Propagation equation for $\mathcal{E}_B$:
  $$-\frac{\partial\overline{\mathcal{E}}_B}{\partial z'} = -\tfrac{1}{2}\alpha_B\overline{\mathcal{E}}_B + \tfrac{1}{2}g_B I_L\overline{\mathcal{E}}_B$$
  - ▷ Attenuation coefficient: $$\alpha_B = \frac{4\pi\omega_B\sigma}{|k_B|c^2}$$
- **Peak SBS gain** (cgs units):

$$\boxed{g_B = \frac{\gamma^2 k_{L,\mathrm{vac}}^2}{c\overline{\rho}c_s\Gamma_B}}$$

where $$k_{L,\mathrm{vac}} = 2\pi/\lambda_{L,\mathrm{vac}}$$

- At $\lambda_{L,\mathrm{vac}} = 1550$ nm, the peak SBS gain in silica fiber is

$$g_B = 4\times 10^{-9}\,\mathrm{cm/W}$$

ELECTROMAGNETIC EQUATIONS FOR SBS IN FIBERS (4)

- Equation for $\overline{\mathcal{E}}_B$ using full expression for acoustic amplitude $q_{k,0}$:

$$-\frac{\partial}{\partial z'}\overline{\mathcal{E}}_B = -\frac{\alpha_B}{2}\overline{\mathcal{E}}_B + \left(\frac{\gamma^2 k\omega_B^2}{64\pi|k_B|c^2\overline{\rho}c_s}\right)\frac{(\Gamma_B/2) + i(c_s k - \omega)}{(c_s k - \omega)^2 + (\Gamma_B/2)^2}|\overline{\mathcal{E}}_L|^2\overline{\mathcal{E}}_B$$

  ▷ The imaginary part of the nonlinear term contributes to the nonlinear refractive index

- **SBS gain** as a function of ω:

$$\boxed{g_B(\omega) = g_B(0)\,\frac{(\Gamma_B/2)^2}{(\omega_s - \omega)^2 + (\Gamma_B/2)^2}}$$

$$g_B(0) = \frac{\gamma^2 k_{L,\text{vac}}^2}{c\overline{\rho}c_s\Gamma_B}, \quad \omega = \omega_L - \omega_B$$

- **SBS frequency shift**:

$$\omega_s = c_s k = 2n_0 c_s k_{L,\text{vac}}$$

$$\frac{\omega_s}{2\pi} = \frac{2n_0 c_s}{\lambda_{L,\text{vac}}} \approx 11\,\text{GHz at } \lambda_{L,\text{vac}} = 1550\,\text{nm}$$

MODULATED PUMPING OF SBS (1)

- Laser and Brillouin fields:

$$\mathbf{E}_L = \hat{\mathbf{x}}\,\mathrm{Re}\left[e^{i(\beta_L z - \omega_{L,0} t)} \sum_{m=0}^{N-1} \mathcal{E}_{L,m} e^{im\Delta\omega(n_0 z/c - t)}\right],$$

$$\mathbf{E}_B = \hat{\mathbf{x}}\,\mathrm{Re}\left[e^{-i(\beta_B z + \omega_{B,0} t)} \sum_{n=0}^{N-1} \mathcal{E}_{B,n} e^{-in\Delta\omega(n_0 z/c + t)}\right]$$

- Paraxial wave equation for $\overline{\mathcal{E}}_{B,j}$:

$$-\frac{\partial}{\partial z'}\overline{\mathcal{E}}_{B,j} = -\frac{\alpha_B}{2}\overline{\mathcal{E}}_{B,j} + \frac{g_B}{2}\frac{c n_0}{8\pi}\sum_{l,m=0}^{N-1} \frac{(\Gamma_B/2)^2 \overline{\mathcal{E}}_{L,l}\overline{\mathcal{E}}^*_{L,m}\overline{\mathcal{E}}_{B,j-l+m}}{[c_s k - \omega - (l-j)\Delta\omega]^2 + (\Gamma_B/2)^2} \times e^{-2i(m-l)\Delta\omega n_0 z/c}$$

- In lightwave systems, where L is at least several km,

$$\frac{\Delta\omega n_0 L}{c} \gg 1$$

- Since $e^{-2i(m-l)\Delta\omega n_0 z/c}$ oscillates many times over a typical distance L, we keep only the $l = m$ terms

MODULATED PUMPING OF SBS (2)

- Paraxial wave equation for $\overline{\mathcal{E}}_{B,j}$ after discarding rapidly oscillating terms:

$$-\frac{\partial}{\partial z'}\overline{\mathcal{E}}_{B,j} = -\frac{\alpha_B}{2}\overline{\mathcal{E}}_{B,j} + \frac{g_B}{2}\frac{cn_0}{8\pi}\sum_{m=0}^{N-1}\frac{(\Gamma_B/2)^2|\overline{\mathcal{E}}_{L,m}|^2\overline{\mathcal{E}}_{B,j}}{[\omega_s - \omega_{L,m} + \omega_{B,m}]^2 + (\Gamma_B/2)^2}$$

- SBS intensity:

$$I_{B,j}(z) = I_{B,j}(L)\ \exp\left\{-\alpha L + [L_e(L) - L_e(z)]g_B \times \sum_{m=0}^{N-1}\frac{(\Gamma_B/2)^2 I_{L,m}}{(\omega_s - \omega_{L,m} + \omega_{B,m})^2 + (\Gamma_B/2)^2}\right\}$$

- Definitions:

$$\omega_s = \frac{2n_0c_s}{\lambda_{L,\mathrm{vac}}}, \quad L_e(z) = \frac{1 - e^{-\alpha z}}{\alpha}$$

- Reference: E. Lichtman *et al.*, *Journal of Lightwave Technology* **7**, 171–174 (1989)

CONTINUOUS PUMP SPECTRUM (1)

- Spectral decomposition:

$$I_B(z,\,t) = \int_{-\infty}^{\infty} \tilde{I}_B(\omega_B,\,z)\, e^{-i\omega_B t}\, d\omega_B$$

- SBS intensity when the pump spectrum is continuous:

$$\tilde{I}_B(\omega_B,\,z) = \tilde{I}_B(\omega_B,\,L)\,\exp\Big\{-\alpha L + [L_e(L) - L_e(z)]g_B \int_{-\infty}^{\infty} \frac{(\Gamma_B/2)^2 \tilde{I}_L(\omega,\,z)\,d\omega}{(\omega - \omega_s - \omega_B)^2 + (\Gamma_B/2)^2}\Big\}$$

▷ Note convolution of SBS spectrum with pump spectrum

- For a CW or narrow-bandwidth pump, $\tilde{I}_L(\omega,\,0) = I_L\delta(\omega - \omega_{L,0})$, and

$$\tilde{I}_B(\omega_B,\,z) = \tilde{I}_B(\omega_B,\,L)\,\exp\Big\{-\alpha L + [L_e(L) - L_e(z)]g_B I_L \frac{(\Gamma_B/2)^2}{(\omega_{L,0} - \omega_s - \omega_B)^2 + (\Gamma_B/2)^2}\Big\}$$

- If $g_B I_L L_e(L) \gg 1$, then the HWHM of $\tilde{I}_B$ is

$$\Delta\omega_{B,1/2} \approx \left[\frac{\log_e 2}{g_B I_L L_e(L)}\right]^{1/2} \frac{\Gamma_B}{2} < \frac{\Gamma_B}{2} \quad \textbf{(SBS gain narrowing)}$$

CONTINUOUS PUMP SPECTRUM (2)

- For a broad-band pump ($\Gamma_L \gg \Gamma_B$),

$$\tilde{I}_B(\omega_B,\, z) = \tilde{I}_B(\omega_B,\, L)\, \exp\Big\{-\alpha L + \pi[L_e(L) - L_e(z)]g_B \tilde{I}_L(\omega_{B,0} + \omega_s,\, 0)\Gamma_B\Big\}$$

- If the laser spectrum $\tilde{I}_L$ is a Lorentzian, then

$$\tilde{I}_L(\omega_{L,0}, 0) = \frac{I_L}{\pi \Gamma_L}$$

$$\Gamma_L = \text{laser HWHM}$$

- Effective SBS gain with broad-band pumping:

$$\boxed{g_{B,\text{eff}} = \frac{\Gamma_B}{\Gamma_L}\, g_B}$$

BUILDUP OF SBS FROM NOISE (1)

- Propagation equation for SBS power in a single mode:

$$-\frac{\partial P_B}{\partial z} = -\alpha_B P_B(z) + g_B I_L(z) P_B(z)$$

where

$$I_L(z) = I_L(0)e^{-\alpha_L z}$$

Solution:

$$P_B(0) = P_B(L)\exp\left[-\alpha_B L + g_B \int_0^L I_L(z)\,dz\right]$$
$$= P_B(L)e^{-\alpha_B L + g_B I_L(0) L_e(L)}$$

  - ▷ Predicts zero Brillouin power at input ($P_B(0) = 0$) when no Brillouin wave is injected at $z = L$ (*i.e.*, when $P_B(L) = 0$)
  - ▷ Does not take accout of spontaneous emission of light at the Brillouin frequency, ω_B

BUILDUP OF SBS FROM NOISE (2)

- Propagation equation for SBS photon number $N_B \propto P_B$:

$$-\frac{\partial N_B}{\partial z} = -\alpha_B N_B(z) + g_B I_L(z)\big(\underbrace{N_B(z)}_{\text{stimulated}} + \underbrace{1}_{\text{spontaneous}}\big)$$

- Solution to photon number propagation equation:

$$\begin{aligned} N_B(z) &\\ = & \underbrace{g_B \int_z^L I_L(z') \exp\left\{-\alpha_B(z-z') + g_B \int_z^{z'} I_L(z'')\, dz''\right\} dz'}_{\text{amplified spontaneous Brillouin scattering}} \\ & + \underbrace{N_B(L) \exp\left\{\alpha_B(z-L) + g_B \int_z^L I_L(z')\, dz'\right\}}_{\text{amplified injected signal}} \end{aligned}$$

- Photon number at input ($z = 0$) with no injected Brillouin wave at $z = L$ when $\alpha_L L \gg 1$:

$$N_B(0) \approx \frac{e^{g_B I_L(0)/\alpha_L}}{g_B I_L(0)/\alpha_L}$$

- Reference: R. G. Smith, *Applied Optics* **11**, 2489 (1972)

THRESHOLD FOR SBS

- Threshold condition for any nonlinear process (including laser action):

$$\text{nonlinear gain} = \text{linear loss}$$

- SBS threshold condition at z:

$$g_B I_L(z) = \alpha_B$$

where

$$I_L(z) = e^{-\alpha_L z} I_L(0)$$

  - ▷ Define the **threshold length** L_T as the distance over which the SBS gain exceeds the linear loss:

$$g_B I_L(0) e^{-\alpha_L L_T} = \alpha_B$$

  - ▷ Formula for L_T:

$$L_T = \frac{1}{\alpha_L} \ln \left(\frac{g_B I_L(0)}{\alpha_B} \right)$$

BUILDUP OF SBS FROM NOISE (3)

- SBS photon number N_B in one mode at input $(z = 0)$ assuming that
  - ▷ one photon is injected at $L = L_T$
  - ▷ there is no spontaneous Brillouin scattering

  is:

$$N_B(0) = \frac{e^{g_B I_L(0)/\alpha_L} - 1}{g_B I_L(0)/\alpha_L} \approx \frac{e^{g_B I_L(0)/\alpha_L}}{g_B I_L(0)/\alpha_L}$$

- Therefore:

$$\left.\begin{array}{c} N_B(0) \text{ due to} \\ \text{amplified} \\ \text{spontaneous} \\ \text{Brillouin} \\ \text{scattering} \end{array}\right\} = \left\{\begin{array}{c} N_B(0) \text{ calculated} \\ \text{assuming} \\ \text{1 Brillouin photon} \\ \text{per mode} \\ \text{injected at } L = L_T \end{array}\right.$$

CRITICAL POWER FOR SBS IN FIBERS (1)

- When the pump power equals the critical power P_{crit},

$$\text{SBS power} = \text{pump power}$$

  at the input ($z = 0$)

  ▷ When $P_L \geq P_{\text{crit}}$, SBS is strong enough to deplete the signal (pump)

  ▷ **Serious system degradation results**

- Formula for critical power in a fiber of length L (cgs units!):

$$\boxed{P_{\text{crit}} \approx 21 \frac{A_e b}{L_e(L)\, g_B}}$$

- Conditions for validity:

  ▷ Uniform fiber (constant Brillouin shift and linewidth)

  ▷ CW or narrow-band operation

  ▷ No pump depletion by SBS (unachievable in practice)

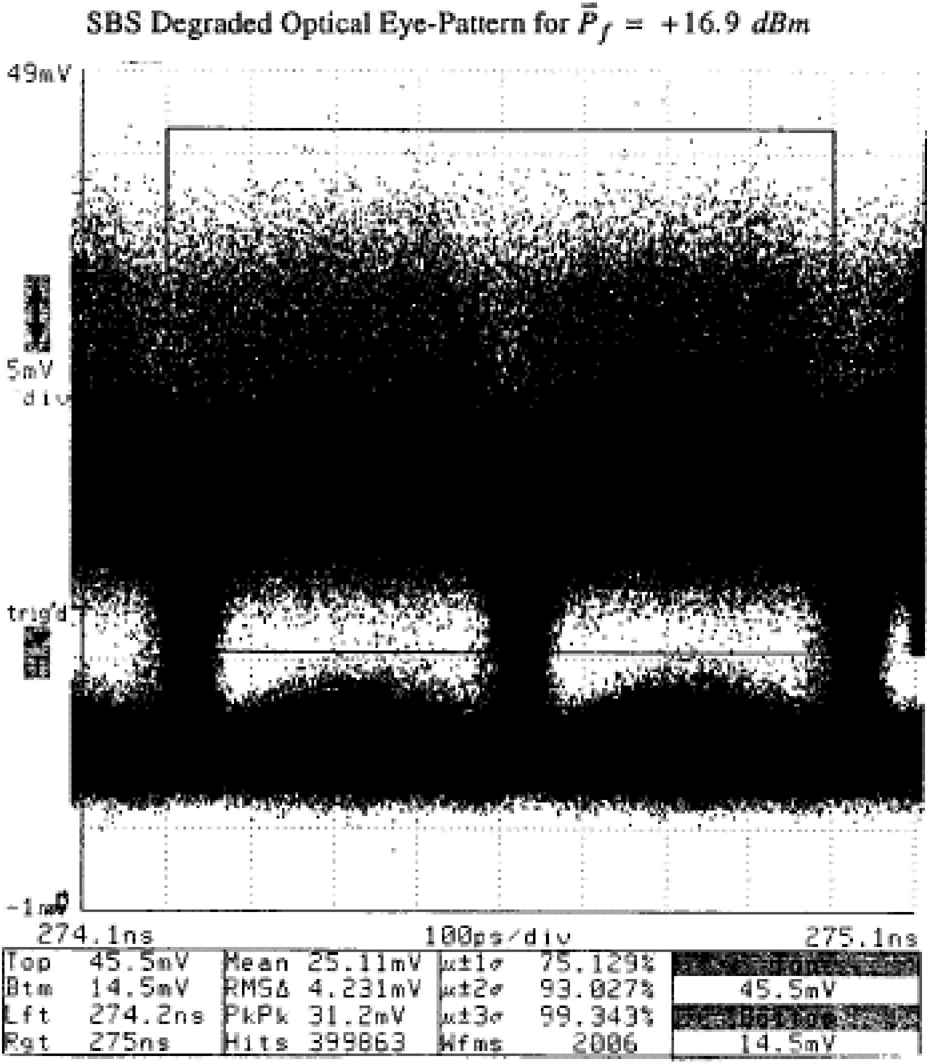


Fig. 4. Typical degraded eye pattern due to SBS for $\bar{P}_f = 16.9$ dB. The signal-to-noise ratio is obtained by measuring the average "on" and "off" voltage levels and then dividing the difference by the sum of the rms fluctuations. In this case the SNR is 3.2 (10.2 dB) which corresponds to a BER floor at 10^{-5}.

CRITICAL POWER FOR SBS IN FIBERS (2)

- In an amplified long-distance telecommunications system, SBS gain must be reduced (critical power increased)
- Increasing the pump (signal) bandwidth decreases the SBS gain:

$$g_B = \frac{\Gamma_B}{\Delta\omega_L} g_{B,CW}$$

Practical ways to increase $\Delta\omega_L$:

- ▷ Dither the center frequency of the laser
- ▷ Modulation at a high bit rate automatically increases the bandwidth
- ▷ Use the non-uniformity of different fiber spools
- ▷ Design fiber with non-constant Brillouin shift

CRITICAL POWER FOR SBS IN FIBERS (3)

- Example 1: CW operation at $\lambda_{L,\text{vac}} = 1550$ nm

$$g_B = 4 \times 10^{-9}\,\text{cm/W} = 4 \times 10^{-16}\,\text{cm} \cdot \text{s/erg}$$

$$A_e = 5 \times 10^{-7}\,\text{cm}^2 \quad (\text{core diam.} = 8\,\mu\text{m})$$

$$\alpha_L = 4.6 \times 10^{-7}\,\text{cm}^{-1} \quad (= .2\,\text{dB/km})$$

$$L \gg L_e \approx 21\,\text{km}$$

$$b = 2 \quad (\text{conventional fiber})$$

$$\boxed{P_{\text{crit}} \approx 2.4\,\text{mW}}$$

- Example 2: Quasi-CW operation in uniform fiber with a dithered laser frequency centered at $\lambda_{L,\text{vac}} = 1550$ nm

$$\frac{\Gamma_B}{\Delta\omega_L} \approx \frac{20\text{–}100\,\text{MHz}}{200\,\text{MHz}} \approx 0.1\text{–}0.5$$

$$\boxed{P_{\text{crit}} \approx 24\,\text{mW–}4.8\,\text{mW}}$$

SBS GAIN FOR AN ASK-MODULATED PUMP (1)

- Laser electric field for NRZ-ASK:

$$\mathcal{E}_{L,ASK}(0,t) = \mathcal{A}\{1 - a[1 - x(t)]\}$$

$$a = 1 - (1 - k_a)^{1/2}$$

$$k_a = \text{modulation depth}$$

$$x(t) = \text{random variable taking values 0 or 1}$$
$$\text{at intervals of } T \text{ (models the signal)}$$

$$B = \text{bit rate} = \frac{1}{T}$$

$$\frac{\text{average ASK laser power}}{\text{CW power}} = \left|\frac{\mathcal{A}}{\mathcal{E}_{L,CW}}\right|^2$$

- SBS gain:

$$g_{B,ASK} = g_{B,CW}\left\{\left(1 - \frac{a}{2}\right)^2 + \frac{a^2}{4}\left[1 - \frac{2B}{\Gamma_B}\left(1 - e^{-\Gamma_B/(2B)}\right)\right]\right\}$$

SBS GAIN FOR AN ASK-MODULATED PUMP (2)

- Assume:
  - ▷ 100% modulation depth:
$$a = 1$$
  - ▷ Average laser power under ASK = CW laser power:
$$\mathcal{A} = \sqrt{2}\,\mathcal{E}_{L,CW}$$
- SBS gain:
$$g_{B,ASK} = g_{B,CW}\left[1 - \frac{B}{\Gamma_B}\left(1 - e^{-\Gamma_B/(2B)}\right)\right]$$
  - ▷ High-bit-rate limit:
$$\frac{\Gamma_B}{B} \ll 1 \;\Rightarrow\; g_{B,ASK} \approx \tfrac{1}{2} g_{B,CW}$$
  - ▷ Critical power in high-bit-rate limit:
$$\boxed{\frac{P_{c,ASK}}{P_c} \approx \frac{1}{2}}$$

"DITHERING" TO SUPPRESS SBS (1)

- Sweep or modulate the carrier frequency ω_L from $\omega_{L,0} - \Omega$ to $\omega_{L,0} + \Omega$, where

$$\Omega \gg \Gamma_B$$

- Gain length l experienced by a backward-propagating SBS wave at frequency ω_B:

$$l \approx \frac{\Gamma_B}{\Omega} L$$

$$L = \frac{c}{n_0} T$$

$$T = \text{sweep time from } \omega_{L,0} - \Omega \text{ to } \omega_{L,0} + \Omega$$

- Optimum:
  - ▷ Sweep time:

$$T \approx \text{transit time across } L_e \approx \frac{n_0 L_e}{c}$$

  - ▷ Dither frequency:

$$\boxed{f_{\text{dither}} \approx \frac{c}{2 n_0 L_e}}$$

"DITHERING" TO SUPPRESS SBS (2)

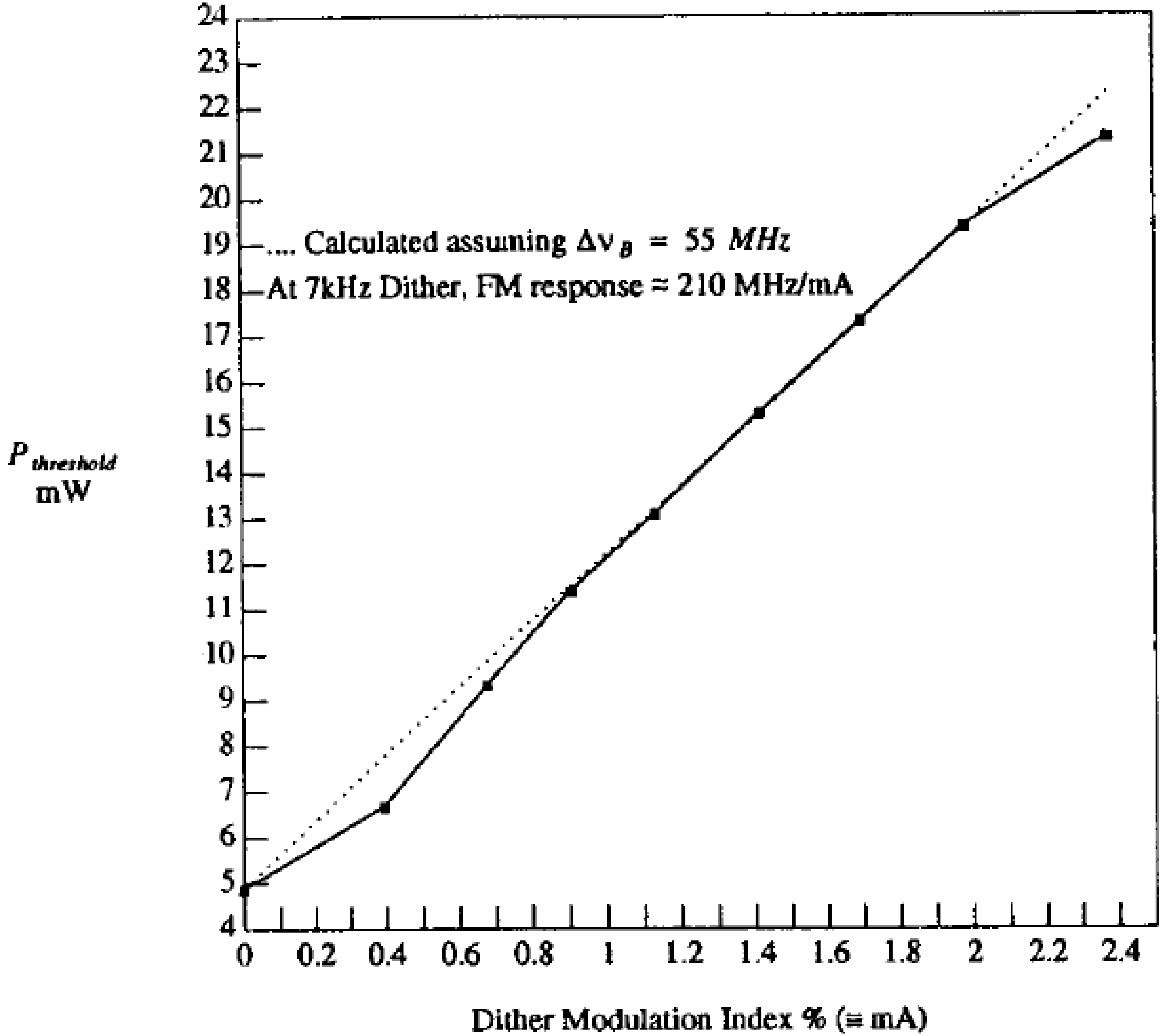
- Assume $L_e \sim 20$ km:

$$\boxed{f_{\text{dither}} \sim 5\ \text{kHz}}$$

- Effectiveness in DSF at $B = 2.5\,\text{Gb}\,\text{s}^{-1}$ and $\lambda_L = 1550$ nm:

Mod. index (%)	$\Omega/2\pi$ (MHz)	$P_{\text{threshold}}$ (mW)
0	0	5
1	210	12
2.5	528	21

- Reference: D. A. Fishman and J. A. Nagel, *Journal of Lightwave Technology* **11**, 1721–1728 (1993)



$P_{threshold}$
mW
.... Calculated assuming $\Delta\nu_B$ = 55 MHz
At 7kHz Dither, FM response ≈ 210 MHz/mA
Dither Modulation Index % (≅ mA)
24
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4