Performance of Optimum Combining with Channel Estimation Errors

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Abstract—**Optimum combining (OC) is an effective way of suppressing interference in receive antenna diversity systems. In this paper, we examine the effect of channel estimation error on bit error probability (BEP) performance of optimum combining with BPSK modulation. The final expression is dependent on channel estimation error variance. Therefore, our analysis is independent of any specific estimation scheme. First, the conditional BEP is derived in terms of eigenvalues of interference and noise covariance matrix. Then, the final BEP expression is obtained using an approximate approach, where each random eigenvalue is replaced by its empirical average over the fading distribution [9]. The numerical results for arbitrary number of antennas and interfering users are presented.**

I. Introduction

Optimum combining is an effective spatial diversity combining technique in multiple user scenarios. It is minimum mean squared error (MMSE) filter applied to the received signal vector from diversity branches. Calculation of optimum combining vector requires knowledge of interference covariance matrix and desired user's channel vector.

In practice, interference covariance matrix as well as desired user's channel is estimated from the finite samples of the received signal. Therefore, estimation errors occur in both interference covariance matrix and desired user's channel. As a result, some performance degradation is observed at the receiver.

Optimum combining reduces to maximal ratio combining for single user scenario. Effect of channel estimation errors are investigated for maximal ratio combiner receivers by several authors. In [1], the output signal-to-noise-ratio (SNR) distribution is expressed in terms of correlation coefficient between the desired user's channel and estimated channel. In [2], receiver performance is evaluated in terms of the variance of the error in channel estimation when training methods are used. The impact of estimation error in covariance matrix is investigated in [3] for sample-matrix-inversion (SMI) method and in [4] for reduced-rank array processing. Both investigations assume perfect knowledge of desired user's channel information.

In this paper, we will assume perfect knowledge of covariance matrix and study the effect of channel estimation error on the BEP performance of optimum combining with BPSK modulation. We extend the work in [5] by expressing the decision variable in terms of estimation error variance and eigenvalues of the interference and noise covariance matrix.

The paper is organized as follows: In section II, our system model is introduced. Section III gives the outline of the procedure to derive the conditional BEP expression. Finally, numerical results are presented in section IV.

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II. System Model

We consider a transmitter with single antenna and a receiver with N-element antenna array. Assuming perfect carrier and time synchronization at the receiver, the received signal $\mathbf{r}$ over N diversity branches with L interfering users at any symbol index is given by

$$\begin{aligned}\mathbf{r} &= \sqrt{P_s}\mathbf{c}_d s_d + \sum_{k=1}^{L}\sqrt{P_k}\mathbf{c}_k s_k + \mathbf{n} \\ &= \sqrt{P_s}\mathbf{c}_d s_d + \mathbf{z}\end{aligned}$$

where $\mathbf{c}_d$ and s_d are the vector channel response and the BPSK symbol of the desired signal respectively. We assume that the desired user's channel vector $\mathbf{c}_d$ and interference and noise vector $\mathbf{z}$ are complex Gaussian distributed with zero mean. Also, independent branches are assumed at the receiver.

Optimum combiner chooses the weight vector to maximize output signal-to-interference-plus-noise-ratio

(SINR). Define the covariance matrix $\mathbf{R} = E\left[\mathbf{z}\mathbf{z}^H\right]$

$$\mathbf{R} = \sum_{k=1}^{L} P_k \mathbf{c}_k \mathbf{c}_k^H + \sigma_n^2 \mathbf{I}_N$$

where the superscript H denotes the Hermitian transposition, σ_n^2 is the power of the noise, and $\mathbf{I}_N$ is an identity matrix of rank N. The optimum combiner vector is $\mathbf{w} = \mathbf{R}^{-1}\hat{\mathbf{c}}_d$ where $\hat{\mathbf{c}}_d$ is the estimated channel vector of the desired user. We can write

$$\hat{\mathbf{c}}_d = \mathbf{c}_d + \mathbf{e}$$

We assume that estimation error vector $\mathbf{e}$ is zero-mean Gaussian distributed with covariance matrix $R_e = \sigma_e^2 \mathbf{I}$. Note that estimation of the desired user signature can be performed first based on a training sequences [2].
Throughout the paper, we will use eigendecomposition of $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$$

where $\mathbf{\Lambda} = \mathrm{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_N\right)$ are eigenvalues of $\mathbf{R}$ in descending order, then $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N_{min}}$ are random and $\lambda_m = \sigma^2$ for $m = N_{min}+1, N_{min}+2, \ldots, N$ where $N_{min} = \min\left(L, N\right)$.

III. Derivation of Bit Error Probability

In this section we derive the BEP of optimum combining conditioned on the eigenvalues of the covariance matrix $\mathbf{R}$. Following the procedure in [5], the decision variable at the input of slicer is expressed as

$$D = \mathbf{w}^H\mathbf{r} + \mathbf{r}^H\mathbf{w} \quad (1)$$

Due to the symmetry of BPSK symbols and given $s_d = +1$, we can calculate the BEP as

$$P_e = \Pr\left(D < 0 \,|\, s_d = +1\right) \quad (2)$$

Using eigendecomposition of $\mathbf{R}$, we can express the decision variable as

$$\begin{aligned} D &= \hat{\mathbf{c}}_d^H \mathbf{R}^{-1}\mathbf{r} + \mathbf{r}^H\mathbf{R}^{-1}\hat{\mathbf{c}}_d \\ &= \hat{\mathbf{c}}_d^H \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^H\mathbf{r} + \mathbf{r}^H\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^H\hat{\mathbf{c}}_d \\ &= \mathbf{x}^H\mathbf{\Lambda}^{-1}\mathbf{y} + \mathbf{y}^H\mathbf{\Lambda}^{-1}\mathbf{x} \\ &= \sum_{m=1}^{M} \lambda_m^{-1}\left(x_m^* y_m + y_m^* x_m\right) \quad (3) \end{aligned}$$

where $\mathbf{x} = \mathbf{U}^H\hat{\mathbf{c}}_d$ and $\mathbf{y} = \mathbf{U}^H\mathbf{r}$. Since $\mathbf{U}$ is unitary, $\mathbf{x}$ is distributed same as $\hat{\mathbf{c}}_d$. Because the BEP is conditioned on eigenvalues, decision variable is quadratic form of complex Gaussian random variables $\mathbf{x}$ and $\mathbf{y}$. It is shown in the appendix that characteristic function for one diversity branch is

$$\phi_m(jw) = \frac{\lambda_m\gamma_m}{\left[w + j\gamma_m\left(\beta_m - \sqrt{P_s}\right)\right]\left[w - j\gamma_m\left(\beta_m + \sqrt{P_s}\right)\right]} \quad (4)$$

where we define

$$\begin{aligned} \beta_m &= \sqrt{(P_s + \lambda_m)(\sigma_e^2 + 1)}, & (5) \\ \gamma_m &= \frac{\lambda_m}{\beta_m^2 - P_s}, & (6) \\ \beta_n &= \sqrt{(P_s + \sigma_n^2)(\sigma_e^2 + 1)}, & (7) \\ \gamma_n &= \frac{\sigma_n^2}{\beta_n^2 - P_s}. & (8) \end{aligned}$$

Note that subscript n is used for parameters due to noise eigenvalues, whereas $m = 1, 2, \ldots, N_{min}$ is used for interferer eigenvalues. Then, we can write the characteristic function of the decision variable D as

$$\begin{aligned} \Phi_D(jw) &= \prod_{m=1}^{N} \phi_m(jw) \\ &= \left\{\frac{\sigma_n^2\gamma_n}{\left[w + j\gamma_n\left(\beta_n - \sqrt{P_s}\right)\right]\left[w - j\gamma_n\left(\beta_n + \sqrt{P_s}\right)\right]}\right\}^{N-N_{min}} \\ &\times \left\{\prod_{m=1}^{N_{min}} \frac{\lambda_m\gamma_m}{\left[w + j\gamma_m\left(\beta_m - \sqrt{P_s}\right)\right]\left[w - j\gamma_m\left(\beta_m + \sqrt{P_s}\right)\right]}\right\} \quad (9) \end{aligned}$$

It was shown in [5] that, conditional BEP can be expressed as

$$P(e|\lambda) = -\sum_{Im(w_k > 0)} Res\left[\frac{\Phi_D(jw)}{w}; w_k\right]$$

where the function $Res\left[\frac{\Phi_D(jw)}{w}; w_k\right]$ denotes the residue of $\frac{\Phi_D(jw)}{w}$ at the pole w_k and sum is taken over upper half of the complex plane.
General expression for the conditional bit error rate can be written as [7]

$$\begin{aligned} P(e|\lambda) &= (-1)^{N+1}\sum_{k=1}^{N_{min}} \frac{\lambda_k}{2\beta_k\eta_k} \\ &\times \left\{\frac{\sigma_n^2\gamma_n}{\left[\eta_k + \kappa_n\right]\left[\eta_k - \eta_n\right]}\right\}^{N-N_{min}} \\ &\times \prod_{m=1,\, m\neq k}^{N_{min}} \frac{\lambda_m\gamma_m}{\left[\eta_k + \kappa_m\right]\left[\eta_k - \eta_m\right]} \\ &+ \left(\sigma_n^2\gamma_n\right)^{N-N_{min}} \\ &\times \sum_{l=0}^{N-N_{min}-1} \binom{N - N_{min} + l - 1}{l} \\ &\times \left\{1 + \frac{\left(\eta_n\right)^{N-N_{min}-l}}{2}\left(-\prod_{m=1}^{N_{min}} \lambda_m\right)\right. \\ &\times \sum_{k=1}^{N_{min}} \frac{1}{\beta_k}\left[\frac{\alpha_{k,1}}{\kappa_k\left(\eta_n + \kappa_k\right)^{N-N_{min}-l}}\right. \\ &+ \left.\left.\frac{\alpha_{k,2}}{\eta_k\left(\eta_n - \eta_k\right)^{N-N_{min}-l}}\right]\right\} \\ &\times \frac{1}{\eta_n^{N-N_{min}-l}\left(2\gamma_n\beta_n\right)^{N-N_{min}+l}} \quad (10) \end{aligned}$$

where

$$\alpha_{k,1} = \prod_{m=1,\, m\neq k}^{N_{min}} \frac{\gamma_m}{[\kappa_m - \kappa_k]\,[\eta_m + \kappa_k]}$$
$$\alpha_{k,2} = \prod_{m=1,\, m\neq k}^{N_{min}} \frac{\gamma_m}{[\eta_m - \eta_k]\,[\kappa_m + \eta_k]}$$

and

$$\eta_m = \frac{\lambda_m}{(\beta_m - \sqrt{P_s})}$$
$$\kappa_m = \frac{\lambda_m}{(\beta_m + \sqrt{P_s})}$$

Note that when there are no channel estimation errors (i.e., $\sigma_e^2 = 0$), we obtain

$$\begin{aligned}\beta_m &= \sqrt{P_s + \lambda_m}\\ \beta_n &= \sqrt{P_s + \sigma_n^2}\\ \gamma_m &= \gamma_n = 1\\ \eta_m &= \sqrt{P_s + \lambda_m} + \sqrt{P_s}\\ \kappa_m &= \sqrt{P_s + \lambda_m} - \sqrt{P_s}\end{aligned}$$

also

$$\alpha_{k,1} = \alpha_{k,2} = \frac{1}{\lambda_m - \lambda_k}\,.$$

Therefore equation (10) reduces to [5, eqn. 10].

When the number of interferers is equal to or more than the number of antennas, i.e. $N_{min} = N$, second sum in equation (10) becomes zeros. Therefore, the conditional BEP equation reduces to

$$P(e|\lambda) = (-1)^{N+1} \sum_{k=1}^{N} \frac{\lambda_k}{2\beta_k \eta_k} \times \prod_{m=1,\, m\neq k}^{N} \frac{\lambda_m \gamma_m}{[\eta_k + \kappa_m]\,[\eta_k - \eta_m]}$$

Training Based Methods

Conditional error probability in (10) is expressed in terms of estimation error variance σ_e, without any assumption on the estimation method. If training based estimation is used, σ_e is dependent on the channel SNR [2], [8]. Therefore, for training methods we can write

$$\sigma_e = \nu \frac{\sigma_n^2}{P_s} = \frac{\nu}{SNR}$$

where ν is some constant depending on the training data size.

Approximation for Unconditional Bit Error Probability

Unconditional BEP can be found from

$$P_e = \int \ldots \int P(e|\lambda) p_\lambda(\lambda) d\lambda \qquad (11)$$

where integration is over eigenvalues resulting from interference, i. e. $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_{min}}\}$. Unfortunately, this integral does not yield an analytically tractable solution. For the sake of analysis, we will approximate the unconditional BEP as

$$P_e \simeq P(e|\tilde{\lambda}) \qquad (12)$$

where $\tilde{\lambda} = E[\lambda]$. This approximation is first order Taylor series expansion of conditional BEP function $P(e|\lambda)$. Similar approximation is made in [9] and [10]. Evaluation of the expectation $E[\lambda]$ still requires Monte Carlo simulations. Some results are presented in [11] to find the expected values of the eigenvalues if number of interferers is two. It will be shown with simulations that this approximation gives very close results in both underloaded and overloaded conditions.

IV. Numerical Examples

In our simulations we evaluate the equation (11) by averaging conditional error probability in the equation (10) over the distribution of eigenvalues. Eigenvalues of the covariance matrix are calculated for each realization of the channel and conditional BEP is evaluated for those values. Therefore, this calculation gives exact average BEP. Also, validity of the approximation in (12) will be assessed with simulations.

In figures 1 and 2 are shown average BEP of optimum combining with channel estimation errors for different SNR values. Interferers are assumed to be equal power, each having a signal-to-interference ratio (S/I) of 10 dB. In figure 1, the number of interferers is less than the number of antennas, whereas in figure 2, the number of interferers is more than the number of antennas. Exact BEP expression in equation (11) is shown by lines and approximation in equation (12) is by markers. It can be seen that approximation in equation (12) gives very close results to the exact expression in equation (11). As can be seen, impact of SNR is more pronounced in underloaded case where some degrees of freedom are available for diversity combining. As the variance of channel estimation error increases, having higher SNR does not give any performance gain.

In figure 3, we evaluate the exact BEP expression for different channel estimation values. Performance of OC without channel estimation errors is also plotted for reference. It can observed that BEP curves are convex function of SNR, given the estimation error variance. Optimum SNR value giving the minimum BEP decreases with increasing estimation error variance.

Figure 4 shows the performance of training based estimation method. When estimation error is dependent on

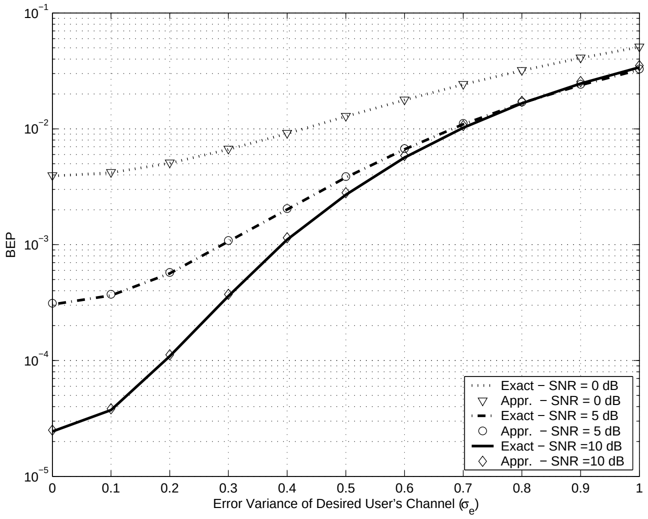

Fig. 1. Average BEP of Optimum Combining with Channel Estimation Errors. $N = 4$, $L = 2$ and $S/I = 10$ dB

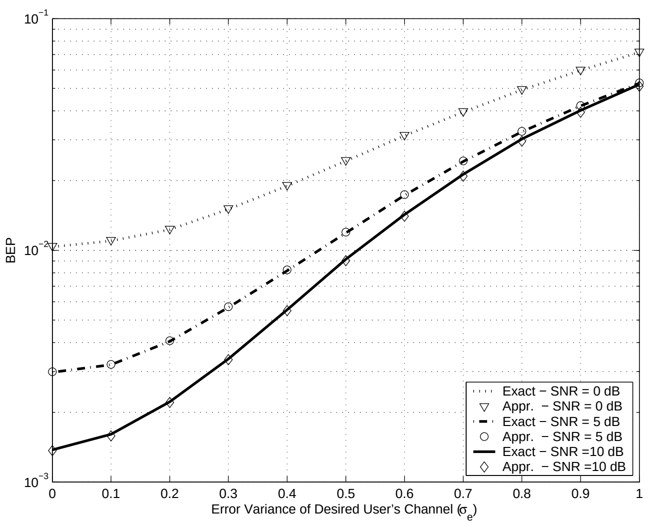

Fig. 2. Average BEP of Optimum Combining with Channel Estimation Errors. $N = 4$, $L = 6$ and $S/I = 10$ dB

the SNR, error floor does not occur, because estimation error reduces as SNR increases. It can be noted that diversity order of the system is preserved even though there is some performace loss. [8].

V. Conclusions

Conditional BEP performance of optimum combining with channel estimation errors has been derived for arbitrary number of interferers under Rayleigh fading. Although the final expression is quite complex, time consuming simulations can be avoided to analyze the performance for different estimation error variances. The expression is not specific to any estimation procedure. Therefore, performance of different channel estimation techniques can be assessed as long as error variance is available.

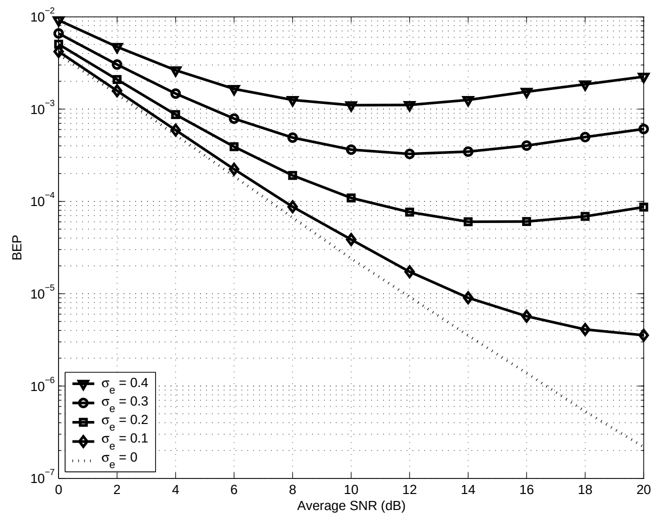

Fig. 3. Average BEP of Optimum Combining with Channel Estimation Errors. $N = 4$, $L = 2$ and $S/I = 10$ dB

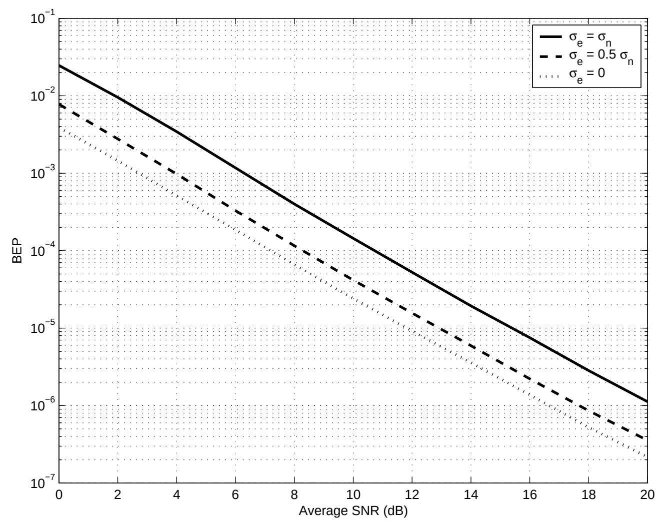

Fig. 4. Average BEP of Optimum Combining with Channel Estimation Errors. $N = 4$, $L = 2$ and $S/I = 10$ dB

VI. Appendix

Derivation of The Characteristic Function of The Decision Variable

From equation (3), we can write the decision variable as

$$D = \sum_{m=1}^{M} \lambda_m^{-1} \left(x_m^* y_m + y_m^* x_m \right) \tag{13}$$

As mentioned this is quadratic form of complex Gaussian variables $\mathbf{x}$ and $\mathbf{y}$. Both vectors have zero mean with covariance matrices

$$\begin{aligned}
\mathbf{R}_{xx} &= E\left[\mathbf{x}\mathbf{x}^H\right] \\
&= E\left[\mathbf{U}^H \hat{\mathbf{c}}_d \hat{\mathbf{c}}_d^H \mathbf{U}\right] \\
&= \mathbf{U}^H E\left[\hat{\mathbf{c}}_d \hat{\mathbf{c}}_d^H\right] \mathbf{U} \\
&= \left(1+\sigma_e^2\right) \mathbf{I}_M \\
\mathbf{R}_{yy} &= E\left[\mathbf{y}\mathbf{y}^H\right] \\
&= E\left[\mathbf{U}^H \mathbf{r}\mathbf{r}^H \mathbf{U}\right] \\
&= \mathbf{U}^H E\left[P_s \mathbf{I}_M + \mathbf{R}\right] \mathbf{U} \\
&= \left(P_s \mathbf{I}_M + \mathbf{\Lambda}\right) \\
\mathbf{R}_{xy} &= E\left[\mathbf{x}\mathbf{y}^H\right] \\
&= E\left[\mathbf{U}^H \hat{\mathbf{c}}_d \mathbf{r}^H \mathbf{U}\right] \\
&= \mathbf{U}^H E\left[\hat{\mathbf{c}}_d \mathbf{r}^H\right] \mathbf{U} \\
&= \sqrt{P_s}\mathbf{I}_M
\end{aligned}$$

As shown in [7] and [5] we can write

$$\begin{aligned}
A &= B = 0 \\
C &= \lambda_m^{-1} \\
w &= \frac{\sqrt{P_s}\lambda_m}{\sigma_e^2 P_s + \left(1+\sigma_e^2\right)\lambda_m} \\
v_1 &= \frac{\lambda_m}{\sqrt{\left(P_s+\lambda_m\right)\left(1+\sigma_e^2\right)} + \sqrt{P_s}} \\
v_2 &= \frac{\lambda_m}{\sqrt{\left(P_s+\lambda_m\right)\left(1+\sigma_e^2\right)} - \sqrt{P_s}}
\end{aligned}$$

Therefore we can write characteristic function of decision variable for single branch as

$$\phi_m(jw) = \frac{\lambda_m \gamma_m}{\left[w + j\gamma_m\left(\beta_m - \sqrt{P_s}\right)\right]\left[w - j\gamma_m\left(\beta_m + \sqrt{P_s}\right)\right]}$$

where we define the β_m and γ_m in equations (5) and (6) respectively.

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