Enumerating pseudo-triangulations in the plane

Sergei Bespamyatnikh*

Abstract

A pseudo-triangle is a simple polygon with exactly three convex vertices. A minimum pseudo-triangulation of a set S is a partition of the convex hull of S into the least number of interior disjoint pseudo-triangles whose vertices are the points of S. The graph of pseudo-triangulations G_{pse} has pseudo-triangulations as vertices and flips as edges. We show that G_{pse} is connected and present an algorithm for enumerating minimum pseudo-triangulations in $O(\log n)$ time per pseudo-triangulation.

1 Introduction

Pseudo-triangles and pseudo-triangulations have received much attention recently because of applications in visibility [8, 9], ray shooting [8], collision detection [7], and rigid motion [5, 12]. There are many open questions related to pseudo-triangulations [13].

A pseudo-triangle is a simple polygon with exactly three convex vertices. For a set S of n points in the plane, a *pseudo-triangulation* T is defined as partition of the convex hull of S into interior disjoint pseudo-triangulations whose vertices are points of S (each point of S is a vertex of T and vice versa). A *minimum pseudo-triangulation* is a pseudo-triangulation with the least number of edges among all pseudo-triangulations of S. Streinu [12] showed that any minimum pseudo-triangulation has $2n-3$ edges. Equivalently, we can define minimum pseudo-triangulation as a pseudo-triangulation with minimum number of pseudo-triangles. Any minimum pseudo-triangulation has $n-2$ pseudo-triangles since $n_f = n_e - n + 1 = n - 2$ by Euler's formula.

*Department of Computer Science, Duke University, Box 90129, Durham, NC 27708, USA. E-mail: besp@cs.duke.edu

Kettner *et al.* [6] proved that every point set in general position has a minimum pseudo-triangulation whose maximum vertex degree is five. This bound is tight. Randall *et al.* [11] gave closed form expressions for the number of triangulations and the number of minimum pseudo-triangulations when S has only one point inside its convex hull. For a point set in general position, they proved an upper bound for pseudo-triangulations. Aichholzer *et al.* [1] establish a comprehensive database for all possible order types of at most 10 points in the plane.

We define the *rank* of a pseudo-triangulation T to be the number of pseudo-triangles in T. One of the operations on the pseudo-triangulation is the *flip*, see Fig. 1. We define the graph of minimum pseudo-triangulations G_{pse} whose vertices correspond to pseudo-triangulations and edges correspond to flips. We show that G_{pse} is connected.

Recently, Brönnimann *et al.* [4] described an algorithm for enumerating all minimum pseudo-triangulations using an efficient technique of Pocchiola and Vegter [10] for finding a flip in $O(1)$ amortized time. Unfortunately the complexity of the algorithm is unknown[1] but the required space is quadratic.

We obtained an algorithm [3] for enumeration of triangulations in $O(\log \log n)$ time per triangulation. Triangulations and minimum pseudo-triangulations can be viewed as pseudo-triangulations of fixed rank (maximum rank for triangulations and minimum rank for minimum pseudo-triangulations). We present an algorithm for enumerating all minimum pseudo-triangulations in $O(\log n)$ time per pseudo-triangulation using flips. In the full paper we generalize two algorithms and give an algorithm for enumerating pseudo-triangulations of a given rank.

[1]The authors conjectured that the running time is $O(\log n)$ per pseudo-triangulation.

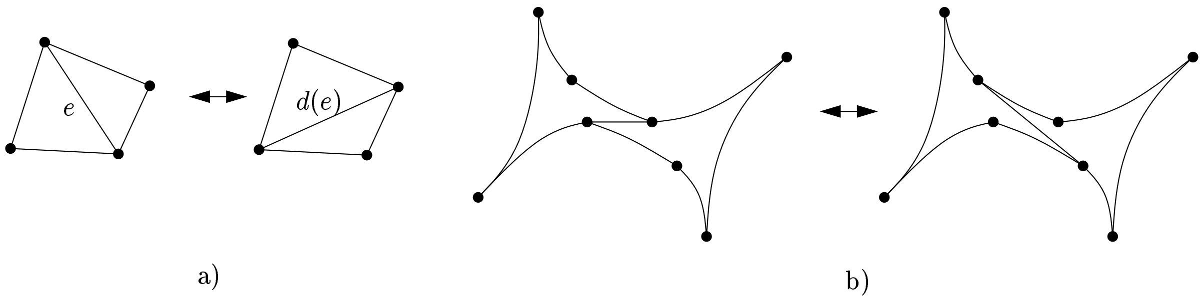

Figure 1: a) Flip of triangles. b) Flip of pseudo-triangles.

2 Preliminaries

Let S be a set of n points. We define a *convex order* of points as follows. Let $S_i = \{p_i, p_{i+1}, \ldots, p_n\}$ for $i = 1, \ldots, n$. An order $(p_1, p_2, \ldots, p_n)$ is *convex* if every point $p_i, i = 1, \ldots, n$ lies on the convex hull of S_i. An example of the convex order is the sorted order by x-coordinates. This order is used in the incremental algorithm for constructing the convex hull. Let $x(p_i)$ and $y(p_i)$ denote the x- and y-coordinate of a point p_i.

A vertex of a polygon is *convex* if incident edges makes angle less than π, otherwise the vertex is *non-convex*. A polygon is called a *pseudo-triangle* if it has exactly three convex vertices. A *pseudo-quadrangle* has exactly four convex vertices. In general, a *pseudo-n-gon* has exactly n convex vertices.

The following lemma generalizes flips in triangulations to flips in pseudo-triangulations, see Fig. 1.

Lemma 1 (Flips in pseudo-quadrangle) *Every pseudo-quadrangle Q has exactly two partitions into two pseudo-triangles that are produced by two diagonals. The diagonals either intersect properly or share a non-convex vertex of pseudo-quadrangle. A diagonal can be computed in $O(\log n)$ time if the vertices of Q are stored in four balanced search trees corresponding to the sides of Q.*

We will call the edge that occurs when an edge e is flipped as the *dual edge of* e, denoted by $d(e)$.

Lemma 2 *Let S be a set of n points whose convex hull is a polygon with n_c, $3 \leq n_c \leq n$ points.*

(i) *The rank of any pseudo-triangulation T of S is in the range $n-2 \leq \mathrm{rank}(T) \leq 2n-n_c-2$.*

(ii) *For any r from the range $n-2 \leq r \leq 2n-n_c-2$, there is a pseudo-triangulation of S with rank r.*

(iii) *A pseudo-triangulation of rank r has $r+n-1$ edges in the range $[2n-3, 3n-n_c-3]$.*

Let $conv(A)$ denote the convex hull of a set A. A pseudo-triangulation T contains all the edges of $conv(S) = conv(S_1)$. We define $index(T)$ as the largest $k \leq n-2$ so that T has the edges of $conv(S_1), \ldots, conv(S_k)$. Clearly, $k \geq 1$ for any pseudo-triangulation.

3 Tree of Minimum Pseudo-Triangulations

In this Section we consider minimum pseudo-triangulations only. Let $(p_1, p_2, \ldots, p_n)$ be a convex order of the points in S. We associate a vector $\alpha(T) = (k, -l)$ with each minimum pseudo-triangulation T. The first number k is the index of T. Thus T has the edges of $conv(S_k)$. l is the number of pseudo-triangles of T that are contained in $conv(S_k)$ and have p_k as the vertex. The lexicographical order of vectors $\alpha(T)$ induces a partial order on the set of pseudo-triangulations. We show later how to make a linear order.

Observation 3 *There is a unique minimum pseudo-triangulation with the lexico-largest vector $\alpha(T) = (n-2, -1)$.*

Proof: T can be obtained by the Henneberg's construction for the point order $(p_n, p_{n-1}, \ldots, p_1)$. ∎

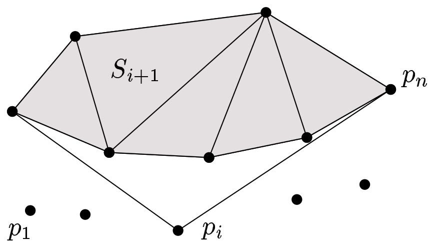

Figure 2: Henneberg's construction.

We denote the pseudo-triangulation with the lexico-largest vector by T_{max}.

Theorem 4 *For any minimum pseudo-triangulation $T \neq T_{max}$, there is a flip making a pseudo-triangulation with lexico-larger vector α.*

Proof: Let $\alpha(T) = (k, -l)$ be the vector of T. Since $\alpha(T)$ is not the lexico-largest vector, $\alpha(T) \neq (n-2, -1)$ and $k < n-2$. The second index l must be greater than 1; otherwise T has all the edges of the convex hull of S_{k+1} and the first index must be less than k. In other words there are at least two pseudo-triangles inside the convex hull of S_k that have the common vertex p_k. Let Δ_1 and Δ_2 be two such pseudo-triangles that share a common edge $p_k p_i$. Let Q be the polygon which is the union of Δ_1 and Δ_2. Our analysis below implies that two pseudo-triangles can share at most one edge.

We show that Q is the pseudo-quadrangle. Note that Q has at least three convex vertices p_k, p_{k_1} and p_{k_2} where p_{k_1} and p_{k_2} are the convex vertices of Δ_1 and Δ_2 different from p_k and p_i, see Fig. 3. On the other hand, there are at most 6 candidates to be convex vertices of Q (convex vertices of Δ_1 and Δ_2). Let γ_1 and γ_2 be the angles between segments incident to p_i in the pseudo-triangle Δ_1 and Δ_2, respectively. We consider the following three cases.

Case 1: $\gamma_1 + \gamma_2 < \pi$. The number of convex vertices of Q is 4 since p_k and p_i are the convex vertices of both Δ_1 and Δ_2, see Fig. 3(a).

Case 2: $\gamma_1 + \gamma_2 > \pi$ and $\gamma_1, \gamma_2 < \pi$. The number of convex vertices of Q is 3, see Fig. 3(b). It contradicts the fact that T is a minimum pseudo-triangulation.

Case 3: $\gamma_1 + \gamma_2 > \pi$ and one of the angles γ_1 or γ_2 is greater than π. The number of convex vertices of Q is 4, see Fig. 3(c).

By Lemma 1 the polygon Q can be partitioned into two pseudo-triangles in two ways and corresponding diagonals do not share vertex p_k. ■

Corollary 5 *G_{pse} is connected and its diameter is at most $(n-1)(n-4)$.*

Proof: The connectivity of G_{pse} follows from the fact that T_{max} can be obtained from any pseudo-triangulation using flips. In order to show the diameter bound it suffices to prove that the length of the partial order on the set of vectors $\alpha(T)$ is at most $(n-1)(n-4)/2$. This follows from the fact that there are $\binom{n-2}{2} = (n-1)(n-4)/2+1$ possible vectors $(k, -l)$.

The vectors $\alpha()$ can be counted as follows. Let k be any integer from $1, \ldots, n-2$ and let T be a pseudo-triangulation of index k. The number of possible pseudo-triangles of T in $conv(S_k)$ incident to p_k ranges from 2 to $n-k-1$. Thus the number of vectors α for a fixed k is $n-k-2$. The total number of vectors $\alpha()$ is

$$\sum_{k=1}^{n-2} n-k-2 = \sum_{i=1}^{n-3} i = \binom{n-2}{2}.$$

■

We show how a rooted tree $\mathcal{T}$ of G_{pse} can be constructed. An approach similar to [3] can be applied to define the tree using the lexicographical order of pseudo-triangulations defined by edge vectors (minimum pseudo-triangulations have a fixed number of edges). This requires the maintenance of an information related to the dual edges incident to a vertex. We apply a different approach for pseudo-triangulations that avoids this problem.

We restrict the order of points to correlate to some direction. For simplicity, we assume that the points are sorted by x-coordinate and their x-coordinates are distinct. The root of $\mathcal{T}$ corresponds to the pseudo-triangulation T_{max}. Let v be a node of G_{pse} such that its pseudo-triangulation $T \neq T_{max}$ and has index $k < n-2$. By Theorem 4 there is a flip of T increasing α. Actually, any edge (p_k, p_i) in the interior of $conv(S_k)$ has this property. Note that $i > k$ and, thus, $x(p_i) > x(p_k)$.

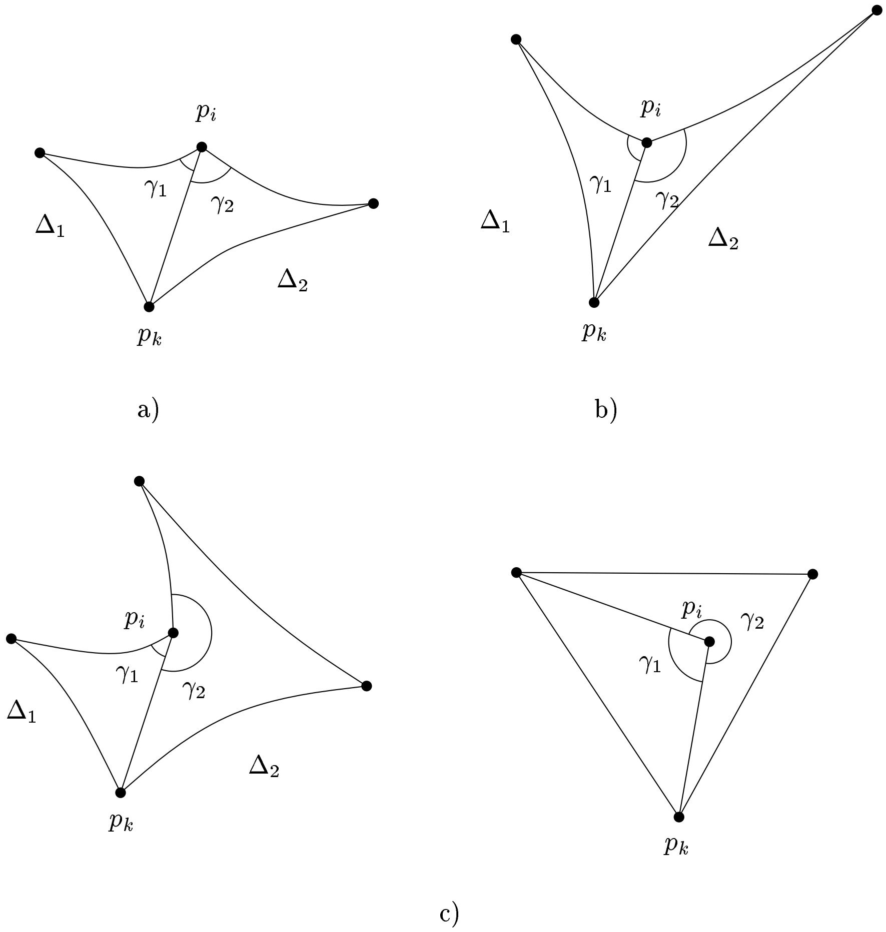

Figure 3: Cases. a) and c) Q is pseudo-quadrangle. b) Q is pseudo-triangle.

Let $e = (p_k, p_i)$ be an interior edge of $conv(S_k)$ such that the slope of $p_k p_i$ is maximum. We define the edge joining v to its parent using the flip of e. Abusing notation, we call e *parent edge* of T and flips leading to the children of v as *child edges*.

Observation 6 (Parent Edge) *An edge $e = (p_a, p_b)$, $a < b$ is the parent edge of a pseudo-triangulation T if and only if $a = index(T)$ and e has the largest slope among all edges of $T - T_{max}$.*

Next we describe the child edges in $\mathcal{T}$. For a pseudo-triangulation T, we define $\Delta(T)$ to be the pseudo-triangle of T that is above the parent edge of T.

Lemma 7 (Child Edge) *Let $(p_k, p_i), a < b$ be the parent edge of a pseudo-triangulation T. Let $p_k p_j p_m$ be the vertices of $\Delta(T)$ in clockwise order and let $p_j = p_{j_1}, p_{j_2}, \ldots, p_m$ be the vertices of the side C of $\Delta(T)$. An edge $e = (p_a, p_b), a < b$ is a child edge of T if and only if e is not an edge of the convex hull of S and one of the following conditions holds*

(i) *$a < k$, or*

(ii) *e is an edge of $conv(S_k)$ but not an edge of $conv(S)$, or*

(iii) *$a = j_r$ and $b = j_{r+1} \neq m$ and both p_a and p_b are visible from p_k in $\Delta(T)$, or*

(iv) (a) *$a = j_r$ and $b = j_{r+1}$, and*

(b) *$(p_a p_b)$ is the last edge of C in counter-clockwise order such that p_a is visible from p_k in $\Delta(T)$, and*

(c) *the dual edge of e is incident to p_k.*

4 Enumerating Minimum Pseudo-Triangulations

The algorithm maintains data structures that allow efficient traversal of $\mathcal{T}$. There are static data structures that store information about order of points and T_{max}. The edges of T_{max} are stored in a binary search tree. For every point p_k we store two edges of $conv(S_k)$ incident to p_k. We store dynamic structures related to a current pseudo-triangulation T. Let L_t be the list of pseudo-triangles of T. For every side of a pseudo-triangle of T, we store the set of its points in a binary search tree in clockwise order. We assume that the following operations can be done in $O(\log n)$ time: search, insertion, deletion, split and concatenate. The edges are joined into sets corresponding to the sides of pseudo-triangles. Every edge occurs in two sets, so we store it twice with pointers to each other. We store

(i) L_e, the list of edges in T and

(ii) L_-, the list of edges of $T - T_{max}$ in lexicographical order. For every $p \in S$ we store L_p the list of edges in $T - T_{max}$ incident to p in sorted order by the slope. The above data structures can be maintained in $O(\log n)$ when a flip is applied.

We apply reverse search technique [2] which can be viewed as the depth-first traversal of the pseudo-triangulation tree $\mathcal{T}$. For the current pseudo-triangulation T, the algorithm maintains its vector $\alpha(T)$, its pseudo-triangle $\Delta(T)$. We traverse $\mathcal{T}$ as follows. The parent edge can be found in $O(\log n)$ time by Lemma 6. The children (the first child and every next child) can be found in $O(\log n)$ time per child using Lemma 7. The visibility query and a flip can be done in $O(\log n)$ time. Therefore we obtain the following theorem.

Theorem 8 *The minimum pseudo-triangulations of a point set in the plane can be reported in $O(\log n)$ time per pseudo-triangulation using linear space.*

References

[1] O. Aichholzer, F. Aurenhammer, and H. Krasser. Enumerating order types for small sets with applications. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pp. 11–18, 2000.

[2] D. Avis and K. Fukuda. Reverse search for enumeration. *Discrete Appl. Math.*, 65:21–46, 1996.

[3] S. Bespamyatnikh. An efficient algorithm for enumeration of triangulations. *Comput. Geom. Theory Appl.*, 2002. to appear.

[4] H. Brönnimann, L. Kettner, M. Pocchiola, and J. Snoeyink. Counting and enumerating pseudo-triangulations with the greedy flip algorithm. In *Fall Workshop on Comput. Geometry*, 2001.

[5] R. Connelly, E. D. Demaine, and G. Rote. Straightening polygonal arcs and convexifying polygonal cycles. In *Proc. 41th Annu. Sympos. on Found. of Computer Science*, pp. 432–442, 2000.

[6] L. Kettner, D. Kirkpatrick, and B. Speckmann. Tight degree bounds for pseudo-triangulations of points. In *Proc. 13th Canad. Conf. Comput. Geom.*, pp. 117–120, 2001.

[7] D. Kirkpatrick, J. Snoeyink, and B. Speckmann. Kinetic collision detection for simple polygons. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pp. 322–330, 2000.

[8] M. Pocchiola and G. Vegter. Pseudo-triangulations: Theory and applications. In *Proc. 12th Annu. ACM Sympos. Comput. Geometry*, pp. 291–300, 1996.

[9] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. *Discrete Comput. Geom.*, 16(4):419–453, 1996.

[10] M. Pocchiola and G. Vegter. The visibility complex. *Int. J. Comput. Geom. Appl.*, 6(3):279–308, 1996.

[11] D. Randall, G. Rote, F. Santos, and J. Snoeyink. Counting triangulations and pseudo-triangulations of wheels. In *Proc. 13th Canad. Conf. Comput. Geom.*, pp. 149–152, 2001.

[12] I. Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *Proc. 41st Annu. IEEE Sympos. Found. Comput. Sci.*, pp. 443–453, 2000.

[13] I. Streinu. Folding carpenter's rulers, robot arms, proteins: a rigidity theoretic approach. *Invited talk, 10th Annual Fall Workshop on Computational Geometry*, 2000.