Equitable relatively-convex partitions of simple polygonal regions (Extended Abstract)

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The two-dimensional case of the discrete *ham-sandwich theorem* (cf. [7]) states that, for finite sets R (red points) and B (blue points) in $\mathbb{R}^2$, there always exists a line L (a *ham-sandwich cut*) such that each of the two open halfplanes defined by L contains at most half of the elements of both R and B. The discrete ham-sandwich problem has attracted considerable attention from an algorithmic viewpoint, culminating in a randomized algorithm by Lo and Steiger [5] with linear expected running time.

Higher dimensional and continuous (with points replaced by general mass distributions) versions of the ham-sandwich problem have also been well studied. Very recently Bose *et al* [2] proposed a new generalization of the discrete ham-sandwich problem in which the sets R and B are restricted to the interior of a simple polygon P and permissible cuts consist of P-*geodesics* (shortest paths in P) joining two boundary points of P. They describe an efficient randomized algorithm for finding a P-geodesic bipartition of P where both of the (open) parts contain at most half of the elements of both R and B. Their algorithm has $O(N \log W)$ expected running time, where $N = |R| + |B| + |P|$ and W measures the non-convexity of P (two plus the number of reflex vertices), which they show is optimal in the algebraic computation tree model.

Suppose that $|R| = kn$ and $|B| = km$, for integers $k \geq 2$, $n, m \geq 0$. A partition of $R \cup B$ into $k \geq 2$ sets is said to be *equitable* if each set contains exact n elements of R and m elements of B. Assuming that the points in R and B are in general position, a ham-sandwich cut provides an equitable bipartition of the plane (i.e. it induces an equitable bipartition of $R \cup B$) into convex regions. Similarly a geodesic ham-sandwich cut provides an equitable bipartition of a polygon P into two relatively-convex regions. (A set $S \subseteq P$ is said to be *convex relative to* P, or simply *relatively-convex* when P is understood, if every P-geodesic joining points of S is contained entirely within S.)

Remark. It is not hard to see that non-simple polygons, even polygons with a single slit in their interior, may not admit equitable bipartitions into relatively convex regions. This precludes the most natural further generalization of ham-sandwich partitions.

In 1998, Kaneko and Kano [4] conjectured that the two-dimensional discrete ham-sandwich theorem can be generalized to the existence of equitable convex k-partitions of the plane (partitions of the plane into k convex regions that induce an equitable partition of $R \cup B$), for any $k \geq 2$. The Kaneko-Kano conjecture was proved independently by Ito *et al* [3], for the case $k = 3$, Sakai [6] and Bespamyatnikh *et al* [1]. (Only [1] considered efficient constructions of equitable convex k-partitions, giving an algorithm with worst-case time $O(N^{4/3} \log^3 N \log k)$, where $N = |R| + |B|$.)

In light of the above, it is natural to ask if there is a analog of the notion of equitable convex k-partitions of the plane, when the sets R and B (as well as partition boundaries) are restricted to the interior of a simple polygon P. Specifically, does P always admit a k-partition into relatively convex regions that induce an equitable partition of $R \cup B$? In this paper we provide an affirmative answer to this question by providing a non-trivial generalization of the construction of Bespamyatnikh *et al* [1]. Specifically, we prove the following:

Theorem 1 *Let $k \geq 2$ and let R and S be point sets, consisting of kn red points and km blue points respectively, in general position in the interior of a simple polygonal region P. Then either (i) there exist*

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integers $0 < k_1, k_2 < k$, *with* $k = k_1 + k_2$, *and a partition of* P *into two relatively-convex regions* P_1, *and* P_2, *where each* P_i *contains exactly* $k_i n$ *red points and* $k_i m$ *blue points, or (ii) there exist integers* $0 < k_1, k_2, k_3 < k$, *with* $k = k_1 + k_2 + k_3$, *and a partition of* P *into three relatively-convex regions* P_1, P_2, *and* P_3, *where each* P_i *contains exactly* $k_i n$ *red points and* $k_i m$ *blue points.*

When $k = 2$ only case (i) is applicable and Theorem 1 is equivalent to the geodesic ham-sandwich (existence) theorem of Bose *et al* [2]. For $k \geq 3$, Theorem 1 can be applied recursively to the regions P_i with $k_i > 1$, resulting in a k-partition of P into relatively convex regions, each of which contains exactly n red and m blue points, as desired. Figure 1 illustrates a relatively-convex equitable 3-partition of a simple polygon containing 9 red (empty) and 3 blue (solid) points.

Remarks. 1. It is not hard to see that (even in the unconstrained setting) partitions of type (i) (reducing bipartitions) do not always exist. For partitions of type (ii), we can assume the partition of P cuts the boundary of P at an arbitrary pre-specified point.

2. Our existence proofs are, in fact, most simply formulated for *continuous* equipartitioning problems (where R and B are replaced by two general mass distributions (measures) on P. In the discrete case, using a natural relaxation of the notion of "equitable", our constructions extend in a direct way to the case when $|R|$ and $|B$ are not necessarily integral multiples of k.

3. The algorithm in [1] for convex equitable k-partitions of the plane generalizes in a natural way to the construction of relatively-convex equitable k-partitions of polygonal regions. A fairly crude implementation uses time (and space) that is (up to logarithmic factors) quadratic in the size of R, B and P. An extended version of this paper will address algorithmic issues in detail.

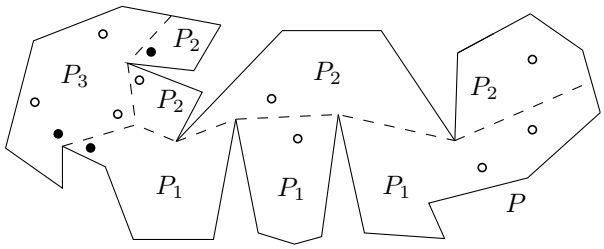

Figure 1: An equitable 3-partition of a polygon.

References

[1] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink. Generalizing ham sandwich cuts to equitable subdivisions. *Discrete Comput. Geom.*, 24(4):605–622, 2000.

[2] P. Bose, E. D. Demaine, F. Hurtado, J. Iacono, S. Langerman, and P. Morin. Geodesic ham-sandwich cuts. In *Proc. 20th Annu. ACM Sympos. Comput. Geom.*, pp. 10–16, 2004.

[3] H. Ito, H. Uehara and M. Yokoyama. 2-dimension ham sandwich theorem for partitioning into three convex pieces. In *Proc. of the Japan Conference on Discrete and Comput. Geometry*, pp. 69–73, 1998.

[4] A. Kaneko and M. Kano. Balanced partitions of two sets of points in the plane, *Comput. Geom. Theory Appl.*, 13:253-261, 1999.

[5] C.-Y. Lo and W. Steiger. An optimal-time algorithm for ham-sandwich cuts in the plane. In *Proc. 2nd Canad. Conf. Comput. Geom.*, pp. 5–9, 1990.

[6] T. Sakai. Radial partitions of point sets in R^2. In *Proc. of the Japan Conference on Discrete and Comput. Geometry*, pp. 74–78, 1998.

[7] R. T. Živaljević. Topological methods. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, pp. 209–224. CRC Press LLC, Boca Raton, FL, 1997.