Stat 5352 Probability & Statistics II

Syllabus

Stat 5352 Course Information

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Text:	Mathematical Statistics with Applications, 7^{rd} Edition
Authors:	Miller and Miller

Topics	**Chapters**
Sampling distributions	8.1-8.8
Point estimation	10.1-10.10
Interval estimation	11.1-11.8
Hypothesis testing	12.1-12.7
Tests for means, variances, proportions	13.1-13.9
Linear regression	14-1-14.5
ANOVA	15.1-15.5

Exam Schedule
To be determined

Grading Policy

Course grade will be based on three exams and homework:
Exam 1, 1/4
Exam 2, 1/4
Exam 3, 1/4
Homework, 1/4.
Note: the complete syllabus is available here:
`http://www.utdallas.edu/~ammann/stat5352_syllabus.pdf`

Homework Assignments

Homework 1

1. Exercise 8.2, p. 272
2. Exercise 8.3, p. 272
3. Exercise 8.21, p. 283
4. Exercise 8.22, p. 283
5. Exercise 8.23, p. 283
6. Exercise 8.24, p. 283
7. Exercise 8.31, p. 284
8. Exercise 8.37, p. 285
9. Exercise 8.64, p. 292

Solutions for Homework 1

1. **Exercise 8.2, p. 272**. Apply Theorem 4.14 with

$$a_i = \begin{cases} \frac{1}{n_1}, & 1 \le i \le n_1, \\ \\ -\frac{1}{n_2}, & n_1 + 1 \le i \le n_1 + n_2. \end{cases}$$

Then

$$Y = \frac{1}{n_1} \sum_{i=1}^{n_1} X_{1i} - \frac{1}{n_1} \sum_{i=1}^{n_2} X_{2i} = \overline{X}_1 - \overline{X}_2,$$

and so $E(Y) = \mu_1 - \mu_2$. Since the r.v.'s are independent, then the covariance term in Theorem 4.14 is 0. Hence,

$$\begin{aligned} Var(\overline{X}_1 - \overline{X}_2) &= \frac{1}{n_1^2} \sum_{i=1}^{n_1} \sigma_1^2 + \frac{1}{n_2^2} \sum_{i=n_1+1}^{n_1+n_2} \sigma_2^2 \\ &= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}. \end{aligned}$$

2. **Exercise 8.3, p. 272**. Let M_1 denote the MGF of X_{11} and let M_2 denote the MGF of X_{21}. Then

$$M_1(t) = \exp\left\{t\mu_1 + t^2\sigma_1^2/2\right\}, \quad M_2(t) = \exp\left\{t\mu_2 + t^2\sigma_2^2/2\right\},$$

and so MGF of Y is

$$\begin{aligned} M_Y(t) &= E\exp\left\{\sum_{i=1}^{n_1}\frac{t}{n_1}X_{1i} - \sum_{i=1}^{n_2}\frac{t}{n_2}X_{2i}\right\} \\ &= [M_1(t/n_1)]^{n_1}[M_2(-t/n_2)]^{n_2} \\ &= \exp\left\{n_1t\mu_1/n_1 + n_1t^2\sigma_1^2/(2n_1^2) - n_2t\mu_2/n_2 + n_2t^2\sigma_2^2/(2n_2^2)\right\} \\ &= \exp\left\{t(\mu_1-\mu_2) + t^2(\sigma_1^2/n_1 + \sigma_2^2/n_2)/2\right\}. \end{aligned}$$

This is the MGF of the normal distribution with mean $\mu_1 - \mu_2$ and variance

$$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

3. **Exercise 8.21, p. 283**. $(n-1)S^2/\sigma^2$ has a chi-square distribution with $n-1$ d.f. This corresponds to a gamma distribution with $\alpha = (n-1)/2$ and $\beta = 2$. The expected value of a gamma r.v. is $\alpha\beta$ and its variance is $\alpha\beta^2$. Therefore,

$$E\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{n-1}{2}2 = n-1,$$

and

$$Var\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{n-1}{2}2^2 = 2(n-1).$$

This implies that

$$E(S^2) = \frac{(n-1)\sigma^2}{n-1} = \sigma^2,$$

and

$$Var(S^2) = 2\frac{(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}.$$

4. **Exercise 8.22, p. 283**. Use Theorem 8.4. Y_n/n is the sample mean of i.i.d. chi-square r.v.'s with 1 d.f., or gamma(1/2,2). So these r.v.'s have expected value 1 and variance 2. Theorem 8.4 implies that the limiting distribution of

$$\frac{Y_n/n - 1}{\sqrt{2/n}}$$

is N(0,1).

5. **Exercise 8.23, p. 283**. Since Y_n has a chi-square distribution with n d.f., then the MGF of $(Y_n - n)/\sqrt{2n}$ is

$$e^{-t\sqrt{n/2}}(1 - t\sqrt{2/n})^{-n/2}.$$

Exercise 8.23 implies that

$$\lim_{n\to\infty} e^{-t\sqrt{n/2}}(1 - t\sqrt{2/n})^{-n/2} = e^{t^2/2}.$$

Let $g(n)$ denote the MGF of $(Y_n - n)/\sqrt{2n}$. Then $g(\nu)$ is the MGF of $(X - \nu)/\sqrt{2\nu}$. Since

$$\lim_{n\to\infty} g(n) = \lim_{\nu\to\infty} g(\nu),$$

then

$$\lim_{\nu\to\infty} e^{-t\sqrt{\nu/2}}(1 - t\sqrt{2/\nu})^{-\nu/2} = e^{t^2/2}.$$

6. **Exercise 8.24, p. 283**.

$$\begin{aligned} P(X > 68) &= P((X - 50)/\sqrt{2(50)} > (68 - 50)/\sqrt{100}) \\ &\approx P(Z > 1.8) \\ &= 0.50 - 0.4641 = 0.0359. \end{aligned}$$

7. **Exercise 8.31, p. 284**. Since the density of the t-distribution is symmetric about 0, then $E(T) = 0$. Also,

$$\begin{aligned} E(T^2) &= k\int_{-\infty}^{\infty} t^2\left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \\ &= 2k\int_0^{\infty} t^2\left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \end{aligned}$$

where k is the constant term in the density function. Now make the substitution

$$\frac{1}{u} = 1 + \frac{t^2}{\nu}.$$

Then

$$t = [\nu(1/u - 1)]^{1/2}$$

and

$$dt = -\frac{1}{2}\nu^{1/2}u^{-3/2}(1 - u)^{-1/2}.$$

Note that

$$(t^2 \to \infty) \Leftrightarrow (u \to 0)$$

and

$$(t^2 \to 0) \Leftrightarrow (u \to 1).$$

Hence,

$$E(T^2) = k\nu^{3/2} \int_0^1 u^{\nu/2-2}(1-u)^{1/2} du.$$

This is a beta integral with $\alpha = (\nu - 2)/2$ and $\beta = 3/2$. Therefore,

$$E(T^2) = k\nu^{3/2} \frac{\Gamma((\nu-2)/2)\Gamma(3/2)}{\Gamma((\nu+1)/2)}.$$

Now use $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ to obtain,

$$\begin{aligned} E(T^2) &= \nu^{3/2} \frac{\Gamma((\nu-2)/2)\Gamma(3/2)}{\Gamma((\nu+1)/2)} \frac{\Gamma((\nu+1)/2)}{\sqrt{v\pi}\Gamma(\nu/2)} \\ &= \frac{\nu}{2} \frac{\Gamma(\nu/2-1)}{\Gamma(\nu/2)} \\ &= \frac{\nu/2}{\nu/2-1} \\ &= \frac{\nu}{\nu-2}. \end{aligned}$$

8. **Exercise 8.37, p. 285**. We have,

   $$Y = \frac{U}{V/\nu_2},$$

   where U has a chi-square distribution with ν_1 d.f. and V has a chi-square distribution with ν_2 d.f. Since

   $$Var(V/\nu_2) = \frac{2\nu_2}{\nu_2^2} = \frac{2}{\nu_2} \to 0$$

   as $\nu_2 \to \infty$, then V/ν_2 converges in probability to $E(V/\nu_2) = 1$. Slutsky's Theorem implies that Y converges in distribution to $U/1 = U$ as $\nu_2 \to \infty$.

9. **Exercise 8.64, p. 292**. We first need to express this interval in terms of the number of s.d.'s from the mean. The s.d. of the sample mean is $6.3/\sqrt{81} = 0.7$, so the interval from 126.6 to 129.4 corresponds to ± 2 s.d.'s from the population mean. Therefore, from Chebychev's Theorem, the probability $\overline{X}$ is outside that interval is at most 0.25 and from the CLT it is approximately $2(0.0228) = .0456$.

Homework 2

1. Exercise 10.2, p. 325
2. Exercise 10.7, p. 326
3. Exercise 10.21, p. 326
4. Exercise 10.23, p. 326
5. Exercise 10.24, p. 326
6. Exercise 10.35, p. 335
7. Exercise 10.38, p. 336

Solutions for Homework 2

1. **Exercise 10.2, p. 325**. $k_1 + k_2 = 1$.
2. **Exercise 10.7, p. 325**.

$$E(\widehat{\theta}) = \left(\frac{X+1}{n+2}\right) = \frac{n\theta + 1}{n+2} = \theta + \frac{1-2\theta}{n+2}.$$

Then, $E(\widehat{\theta}) = \theta$ if and only if $\theta = 1/2$. So $E(\widehat{\theta})$ is not unbiased. However,

$$\lim_{n\to\infty} E(\widehat{\theta}) = \theta,$$

for all θ, so this estimator is asymptotically unbiased.

3. **Exercise 10.21, p. 326**. Since $E(\overline{X}_1) = \mu$ and $E(\overline{X}_2) = \mu$, then

$$E(\omega\overline{X}_1 + (1-\omega)\overline{X}_2) = \omega\mu + (1-\omega)\mu = \mu,$$

so this estimator is unbiased. Also, since the samples are independent, then

$$\begin{aligned} Var(\hat{\mu}) &= Var(\omega\overline{X}_1 + (1-\omega)\overline{X}_2) \\ &= \omega^2 Var(\overline{X}_1) + (1-\omega)^2 Var(\overline{X}_2) \\ &= \omega^2\sigma_1^2/n + (1-\omega)^2\sigma_2^2/n. \end{aligned}$$

Take the derivative wrt ω and set it equal to 0.

$$\begin{aligned} \frac{\partial}{\partial\omega} Var(\hat{\mu}) &= 2\omega\sigma_1^2/n - 2(1-\omega)\sigma_2^2/n \\ &= \frac{2}{n}(\omega(\sigma_1^2 + \sigma_2^2) - \sigma_2^2) \\ &= 0. \end{aligned}$$

Therefore, solution is

$$\omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

Note that the second derivative wrt ω is

$$\frac{2}{n}(\sigma_1^2 + \sigma_2^2) > 0,$$

so this is a minimum.

4. **Exercise 10.23, p. 326**. Similar to Exercise 10.21,

   $$Var(\hat{\mu}) = \omega^2\sigma^2/n_1 + (1-\omega)^2\sigma^2/n_2.$$

   Take derivative wrt ω:

   $$\frac{\partial}{\partial\omega}Var(\hat{\mu}) = 2\sigma^2[\omega(1/n_1 + 1/n_2) - 1/n_2] = 0,$$

   and so

   $$\omega = \frac{1/n_2}{1/n_1 + 1/n_2} = \frac{n_1}{n_1 + n_2}.$$

5. **Exercise 10.24, p. 326**. For $\omega = 1/2$,

   $$Var(\hat{\mu}) = \frac{\sigma^2}{4}(\frac{1}{n_1} + \frac{1}{n_2}),$$

   and for $\omega = n_1/(n_1 + n_2)$,

   $$Var(\hat{\mu}) = \frac{\sigma^2}{n_1 + n_2}.$$

   So efficiency is

   $$\begin{aligned} Eff &= \frac{\frac{\sigma^2}{n_1+n_2}}{\frac{\sigma^2}{4}(\frac{1}{n_1} + \frac{1}{n_2})} \\ &= \frac{4n_1n_2}{(n_1 + n_2)^2}. \end{aligned}$$

   Note that if $n_1 = n_2$, then $Eff = 1$. Otherwise, $Eff < 1$.

6. **Exercise 10.35, p. 335**. The density function for Y_n is

$$g_n(y) = n\beta^{-n}y^{n-1},\ 0 < y < \beta,$$

and so

$$P(Y_n \leq x) = (x/\beta)^n.$$

Let $\epsilon > 0$. Then

$$\begin{aligned} P(|Y_n - \beta| > \epsilon) &= P(Y_n < \beta - \epsilon) \\ &= \left(\frac{\beta - \epsilon}{\beta}\right)^n \\ &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

7. **Exercise 10.38, p. 336**. From Problem 21,

$$\lim_{n\rightarrow\infty} \hat{\mu} = \lim_{n\rightarrow\infty} Var(\omega\overline{X}_1 + (1-\omega)\overline{X}_2) = \lim_{n\rightarrow\infty}(\omega^2\sigma_1^2/n + (1-\omega)^2\sigma_2^2/n) = 0.$$

Hence, $\hat{\mu}$ converges to μ in mean square, which implies by Chebychev's Theorem that $\hat{\mu}$ converges to μ in probability. Therefore, $\hat{\mu}$ is a consistent estimator of μ.

Homework 3

1. Exercise 8.66, p. 292
2. Exercise 8.70, p. 292
3. Exercise 8.71, p. 292
4. Exercise 8.76, p. 293
5. Exercise 8.80, p. 293
6. Exercise 10.6, p. 326
7. Exercise 10.14, p. 326
8. Exercise 10.15, p. 326
9. Show that the estimator in Exercise 10.7, p. 326, is consistent.

Solutions for Homework 3

1. Exercise 8.66, p. 292. Use $N(\mu = 4, \sigma^2 = 16/225)$.

$$\begin{aligned} P(\overline{X} > 4.5) &\approx P(Z > (4.5 - 4) * 15/4) \\ &= P(Z > 1.88) \\ &= .5 - .4699 \\ &= 0.0301. \end{aligned}$$

2. Exercise 8.70, p. 292. $E(\overline{X}_1 - \overline{X}_2) = 0$,

$$SD(\overline{X}_1 - \overline{X}_2) = \sqrt{20^2/400 + 30^2/400} = 1.803.$$

To get probability of 0.99, we need to use $k = 10$ standard deviations.

$$P(|\overline{X}_1 - \overline{X}_2| \leq 18.03) \geq 0.99.$$

3. Exercise 8.71, p. 292. $z = 2.58$, so

$$k = 2.58(1.803) = 4.65.$$

4. Exercise 8.76, p. 293. Need to find $P(s^2 > 54.668) + P(s^2 < 12.102)$. Sampling distribution of $15s^2/25 = .6s^2$ is χ^2 with 15 d.f. so

$$P(s^2 > 54.668) + P(s^2 < 12.102) = P(Y > 32.801) + P(Y < 7.261) = .055.$$

5. Exercise 8.80, p. 293.

$$P(s_1^2/s_2^2 > 1.16) = P(F > 1.16 * 18/12) = P(F > 1.74) = 0.05.$$

6. Exercise 10.6, p. 326.

$$\begin{aligned} E(\overline{X}^2) &= Var(\overline{X}) + (E(\overline{X}))^2 \\ &= \sigma^2/n + \mu^2 \\ &\rightarrow \mu^2 \end{aligned}$$

as $n \rightarrow \infty$.

7. Exercise 10.14, p. 326. Neglecting constants,

$$l(\theta) = log(f(y;\theta)) = y \log(\theta) + (1-y)\log(1-\theta),$$

$$\begin{aligned} \frac{\partial}{\partial\theta} l(\theta) &= \frac{y}{\theta} - \frac{1-y}{1-\theta} \\ &= \frac{x-\theta}{\theta(1-\theta)}. \end{aligned}$$

Fisher's information for Bernoulli distribution is

$$E\left(\frac{Y-\theta}{\theta(1-\theta)}\right)^2 = (\theta(1-\theta))^{-1},$$

so C-R lower bound for variance is $\theta(1-\theta)/n = Var(X/n)$.

8. Exercise 10.15, p. 326. Neglecting constants,

$$l(\lambda) = log(f(y;\lambda)) = y \log(\lambda) - \lambda.$$

$$\begin{aligned} \frac{\partial}{\partial\lambda} l(\lambda) &= \frac{y}{\lambda} - 1 \\ &= \frac{y-\lambda}{\lambda}. \end{aligned}$$

Fisher's information for Poisson distribution is

$$E\left(\frac{Y-\lambda}{\lambda}\right)^2 = \lambda^{-1}$$

so C-R lower bound is $\lambda/n = Var(\overline{X})$. Since the sample mean is unbiased for Poisson, then it is MVUE.

9.

$$Var\left(\frac{X+1}{n+2}\right) = \frac{Var(X)}{(n+2)^2} = \frac{n\theta(1-\theta)}{(n+2)^2},$$

which converges to 0 as $n \rightarrow \infty$, so this estimator is consistent.

Homework 4

1. Experience has shown that the mean number of sick leave days taken per year by employees in a large corporation is 10 with a standard deviation of 4. Suppose that a random sample of 62 employees who smoke is selected, and suppose that their sick leave records for the last year show that the average number of sick leave days taken is 15 with a s.d. of 12.

   (a) Construct a 95% confidence interval for the mean number of sick leave days for all employees who smoke.

   (b) Construct a 95% confidence interval for the standard deviation of the number of sick leave days for all employees who smoke.

   (c) What can you say about the sick leave taken by employees who smoke compared to the overall number of sick leave days?

2. A researcher is interested in the long-term results of individuals who use the weight-loss center at a particular private hospital. In particular, she would like to know if there has been any change in the mean weight of these individuals 1 year after finishing the program. Suppose that she randomly selects 25 of these individuals and finds that the mean weight loss after 1 year is 5.4 with a s.d. of 8.2 (a negative loss represents a weight gain). You may assume that the distribution of weight loss is approximately a normal distribution.

   (a) Construct a 90% confidence interval for the mean weight loss after 1 year. What conclusions can you make?

   (b) Construct a 90% confidence interval for the standard deviation of weight loss after 1 year.

   (c) Use the information in this sample to determine the sample size required to estimate the mean weight loss after 1 year to within 1 pound with 95% confidence.

3. A large corporation requires that its employees attend a 1-day sexual harassment seminar. The Director of Human Resources of this corporation would like to determine whether or not the information presented in this seminar is retained over a long period of time. To this end, a random sample of 30 employees is randomly selected from recently hired employees who are scheduled to take this seminar. Each of the employees in this sample completes a test of knowledge concerning sexual harassment and related legal issues immediately after the seminar, and then takes a similar test 6 months later. The scores are summarized below. Construct a 95% confidence interval for the difference between the mean scores.
   Mean(first test) = 83.6, s.d.(first test) = 12.4
   mean(second test) = 74.8, s.d.(second test) = 15.8
   s.d.(differences) = 18.1.

4. A corporation would like to convert its training courses from instructor-led classroom courses to individual computer instruction to reduce costs. To determine if there is a difference in performance of these two training methods, a sample of 30 new employees was randomly assigned to two groups of 15 each. The first group received the standard classroom training and the second group received the individual computer instruction. After completing the training, all students took the post-training exam with the following results:
   $\bar{X}_1 = 84.4$, $s_1 = 12.5$, $\bar{X}_2 = 79.1$, $s_2 = 10.6$.
   Construct a 90% confidence interval for the difference between the mean scores of the two groups. You may assume that the histograms of the test scores are approximately normal curves.

5. A researcher wishes to estimate with 90% confidence the proportion of employees in a particular industry who favor random drug testing of all employees in this industry. She also would like to have her estimate be within 0.02 of the true value with this level of confidence. What sample size should she use if she is not willing to place any prior bounds on this proportion? What sample size should she use if she believes that this proportion will be at least 75%?

6. Suppose that the mean IQ scores of 12 year olds with a particular learning disability is 72, and you wish to determine if a new protocol for educating these students can improve their IQ. A random sample of 24 such children receive this new protocol, and are then tested 1 month after completion. Suppose that these results show a mean IQ for these 24 students is 75 with a standard deviation of 12.

   (a) Construct a 95% confidence interval for the mean IQ of students who receive this protocol. You may assume that the IQ scores have approximately a normal distribution.
   (b) Next use this information as a preliminary sample to determine the sample size necessary to estimate the mean of this population to within 3 points with 95% confidence.

7. A survey of 600 owners of 2010 Ford Explorers showed that 375 of these owners were satisfied with the reliability of this vehicle. Construct a 90% confidence interval for the proportion of all such owners who are satisfied with the reliability of this car model. What sample size would be necessary to estimate this proportion to within .02 with 95% confidence if no assumptions are made regarding this population proportion?

8. Suppose that the mean time to complete a certain task has approximately a normal distribution with mean of 8.3 minutes and s.d. 1.8. After rearranging the order of some of the tasks that make up the job, it was found that the mean completion time of a random sample of 18 such jobs was 7.6 minutes with a s.d. of 1.2 minutes. You may assume that these times are approximately normally distributed. Construct a

90% confidence interval for the mean completion time after the rearrangement and for the s.d. of the completion time. What conclusions can you make?

9. A large accounting firm would like to estimate the proportion of errors made by its employees on tax filings. It randomly selects 250 files and after close examination finds that 20 contain errors.
   (a) Construct a 90% confidence interval for the overall proportion of errors made on its tax filings. What sample size would be necessary to estimate this proportion to within 0.04 with 90% confidence if the firm is willing to assume that this proportion should be no more than 12%?
   (b) Now suppose that the total dollar value of the errors on those 20 files averaged \$425 with a standard deviation of \$80. Assume that the histogram of the dollar value of errors is approximately a normal curve and treat the 20 erroneous files as a random sample of all erroneous files. Construct a 90% confidence interval for the mean dollar value
   (c) and a 90% confidence interval for the standard deviation of the dollar value of errors on all erroneous files.

Solutions for Homework 4

1. (a) $d.f. = 61$, so use bottom row of t-table: $t = 1.96$. Confidence interval for mean is

$$15 \pm 1.96\frac{12}{\sqrt{62}} \Leftrightarrow 15 \pm 3.0 \Leftrightarrow [12, 18].$$

(b) We can use normal approximation to chisquare to construct confidence interval for variance.

$$\begin{aligned} C_L &\approx (n-1) - 1.96\sqrt{2(n-1)} = 61 - 21.65 = 39.35, \\ C_U &\approx (n-1) + 1.96\sqrt{2(n-1)} = 61 + 21.65 = 82.65. \end{aligned}$$

Note: actual values are: $C_L = 41.3$, $C_U = 84.48$. This gives the confidence interval:

$$[12\sqrt{61/82.65}, 12\sqrt{61/39.35}] \Leftrightarrow [10.31, 14.94].$$

(c) Since all of the values in the 95% confidence interval for the mean are greater than the overall mean, we are at least 95% confident that the mean for smokers is greater than the overall mean.

2. (a) $d.f. = 24$, $t = 1.71$. Confidence interval for mean is

$$5.4 \pm (1.711)(8.2)/\sqrt{25} \Leftrightarrow 5.4 \pm 2.81 \Leftrightarrow [2.59, 8.21].$$

All of these values are greater than 0, so we are at least 90% confident that the mean weight loss one year after the program is still greater than 0.

(b) $C_L = 13.85$, $C_U = 36.42$. Confidence interval is

$$[8.2\sqrt{24/36.42}, 8.2\sqrt{24/13.85}] \Leftrightarrow [6.66, 10.79].$$

(c) $n = (1.96 * 8.2/1)^2 = 258$.

3. Paired samples. $d.f. = 29, t = 2.045$. Confidence interval for difference between means is

$$(83.6 - 74.8) \pm (2.045)(18.1)/\sqrt{30} \Leftrightarrow 8.8 \pm 6.76 \Leftrightarrow [2.04, 15.56].$$

4. Independent samples.

$$\begin{aligned} V_1 &= 12.5^2/15 = 10.42, \quad V_2 = 10.6^2/15 = 7.49 \\ df &= \frac{(10.42 + 7.49)^2}{(10.42^2/14) + (7.49^2/14)} = 27 \\ t &= 1.70. \end{aligned}$$

Confidence interval for difference between means is

$$(84.4 - 79.1) \pm (1.70)\sqrt{10.42 + 7.49} \Leftrightarrow 5.3 \pm 7.19 \Leftrightarrow [-1.89, 12.49].$$

5. With no prior bound: $n = (1.645/.04)^2 = 1691$. With prior bound of 0.75, $n = (1.645/.02)^2(.75)(.25) = 1268$.

6. (a)

$$75 \pm (2.069)(12)/\sqrt{24} \Leftrightarrow 75 \pm 5.07 \Leftrightarrow [69.93, 80.07]$$

   (b) $n = ((1.96)(12)/3)^2 = 61$.

7. $\hat{p} = 0.625$. Confidence interval is

$$.625 \pm 1.645\sqrt{(.625)(.375)/600} \Leftrightarrow .625 \pm .033 \Leftrightarrow [.592, .658].$$

   $n = (1.96/.04)^2 = 2401$.

8. Confidence interval for mean:

$$7.6 \pm (1.74)(1.2/\sqrt{18} \Leftrightarrow 7.6 \pm 0.49 \Leftrightarrow [7.11, 8.09].$$

   Confidence interval for s.d.:

$$[1.2\sqrt{17/27.59}, 1.2\sqrt{17/8.67}] \Leftrightarrow [0.94, 1.68].$$

   All reasonable mean values are less than 8.3, so rearrangement has lower mean time. All reasonable values for the s.d. are below 1.8, so rearrangement also has reduced s.d.

9. (a)

$$.08 \pm 1.645\sqrt{(.08)(.92)/250} \Leftrightarrow .08 \pm .028 \Leftrightarrow [.052, .108].$$

   Sample size is

$$n = (1.645/.04)^2(.12)(.88) = 179.$$

   (b)

$$425 \pm (1.729)(80)/\sqrt{20} \Leftrightarrow 425 \pm 30.9 \Leftrightarrow [394.1, 455.9].$$

   (c)

$$[80\sqrt{19/30.14}, 80\sqrt{19/10.12}] \Leftrightarrow [63.5, 109.6].$$

Homework 5

Note: all hypothesis testing problems should include a p-value as part of your answer.

1. A large corporation requires that its employees attend a 1-day sexual harassment seminar. The Director of Human Resources of this corporation would like to determine if the information presented in this seminar is retained over a long period of time. To this end, a random sample of 30 employees is randomly selected from recently hired employees who are scheduled to take this seminar. Each of the employees in this sample completes a test of knowledge concerning sexual harassment and related legal issues immediately after the seminar, and then takes a similar test 6 months later. The scores are summarized below.
   Mean(first test) = 83.6, s.d.(first test) = 12.4
   mean(second test) = 74.8, s.d.(second test) = 15.8
   s.d.(differences) = 18.1. What decision can you make at the 10% level of significance?

2. A corporation would like to convert its training courses from instructor-led classroom courses to individual computer instruction to reduce costs. A random sample of 30 new employees was randomly assigned to two groups of 15 each. The first group received the standard classroom training (CC) and the second group received the individual computer instruction (CI). After completing the training, all students took the post-training exam with the following results:
   $\bar{X}_{CC} = 84.4$, $s_{CC} = 12.5$, $\bar{X}_{CI} = 79.1$, $s_{CI} = 10.6$.
   a. The corporation will convert to CI unless there is evidence at the 5% level of significance that the mean score of of employees who receive CI is lower than those who received CC. Should the corporation convert?
   b. The corporation also would like to know if the standard deviation of scores in CI is less than the s.d. of scores in CC. What decision can you make at the 5% level of significance?

3. The overall owner satisfaction of all 2010 domestic SUV's is 56%. A survey of 600 owners of 2010 Ford Explorers was conducted to determine if the owner satisfaction rate for these SUV's differs from the overall rate. Suppose 375 of those surveyed were satisfied with their Explorer.
   a. What conclusion can you make at the 10% level of significance?
   b. What is the probability of rejecting the null hypothesis if the population proportion for this group is actually 0.70?

4. Suppose that the mean time to complete a certain task has approximately a normal distribution with mean of 8.3 minutes and s.d. 1.8. After rearranging the order of subtasks that make up the job, it was found that the mean completion time of a random sample of 18 such jobs was 7.6 minutes with a s.d. of 1.2 minutes. You may assume that these times are approximately normally distributed. Has this rearrangement reduced the mean task time at the 1% level of significance? What is the p-value of this data?

5. It has been claimed by those who have developed the SAT that special courses for this exam have no effect on the overall scores. Suppose that you are interested in this question, and specifically would like to determine if such a course that is advertised locally has an effect, good or bad, on SAT scores. You arrange to have a random sample of 40 students take this course before the SAT is given. Suppose that the average score for all students who took the exam at that time was 440 on the verbal portion and 425 on the quantitative portion. Also, suppose that the sample of 40 students who took the course had a mean verbal score of 452 with a s.d. of 110, and a mean quantitative score of 448 with a s.d. of 80. What conclusions can you make at the 10% level of significance? Which type errors might you make with these decisions? Construct 95% confidence intervals for the mean Verbal and mean Quantitative scores for students who take this course.

6. Suppose initial quality scores used by an automobile manufacturer for its luxury brand SUV are approximately normally distributed. Suppose also that this manufacturer has standards for both the mean and s.d. of initial quality scores given by: mean score should be at least 90 and the s.d. should be no more than 3. A random sample of 20 of these SUVs is selected from a week's production run and tested for initial quality. It was found that the mean score was 88 and the s.d. was 5.2. Does this data show at the 5% level of significance that the standards are not being met?

Solutions for Homework 5

1. This is paired data. Because of the ambiguous wording, I would accept whichever hypotheses you used. If you used

   $$H_0: \ \mu_1 > \mu_2 \quad H_1: \ \mu_1 \leq \mu_2$$

   Test statistic is

   $$\frac{74.8 - 83.6}{18.1/\sqrt{30}} = -2.663$$

   p-value is greater than 0.50 so do not reject the null hypothesis.

   $$H_0: \ \mu_1 \leq \mu_2 \quad H_1: \ \mu_1 > \mu_2$$

   Test statistic is

   $$\frac{83.6 - 74.8}{18.1/\sqrt{30}} = 2.663$$

   p-value is between 0.005 and 0.01 so reject the null hypothesis.
   For two-sided alternative, p-value is between 0.01 and 0.02 so reject the null hypothesis.

2. **a**. Independent samples.

   $$H_0: \ \mu_{CC} \leq \mu_{CI} \quad H_1: \ \mu_{CC} > \mu_{CI}$$

   $V_{CC} = 12.5^2/15 = 10.42$, $V_{CI} = 10.6^2/15 = 7.49$,

   $$df = \frac{(10.42 + 7.49)^2}{10.42^2/14 + 7.49^2/14} = 27$$

   Test statistic is

   $$\frac{84.4 - 79.1}{\sqrt{10..42 + 7.49}} = 1.25$$

   P-value is greater than 0.10, so do not reject null hypothesis. The company should convert to CI.
   b. $H_0: \ \sigma^2_{cc}/\sigma^2_{ci} \leq 1 \quad H_1: \ \sigma^2_{cc}/\sigma^2_{ci} > 1$. Test statistic is $s^2_{cc}/s^2_{ci} = 1.39$, p-value comes from $F_{14,14}$. Since the critical value for $\alpha = .05$ from the F-table is 2.48, the $\mathrm{p-value} > .05$, so do not reject H_0.

3. **a**.

   $$H_0: \ \pi = 0.56 \quad H_1: \ \mu \neq 0.56$$

$\hat{p} = 375/600 = .625$. P-value is

$$P(|Z| \geq \frac{.625 - .56}{\sqrt{(.56)(.44)/600}}) = P(|Z| \geq 3.21) = 2P(Z \geq 3.21) < 0.002,$$

so reject $H_)$.
b. First obtain the rejection region. Critical value from $N(0,1)$ for 2-sided test with $\alpha = .10$ is 1.645. Rejection region is then

$$(\hat{p} \geq .56 + 1.645\sqrt{(.56)(.44)/600}) \cup (\hat{p} \leq .56 - 1.645\sqrt{(.56)(.44)/600}),$$

$$(\hat{p} \geq 0.593) \cup (\hat{p} \leq 0.527).$$

Power function at 0.70 is then

$$\begin{aligned} Power(.7) &= P(Z \geq \frac{.593 - .70}{\sqrt{(.7)(.3)/600}} + P(Z \leq \frac{.527 - .70}{\sqrt{(.7)(.3)/600}} \\ &= P(Z \geq -4.67) + P(Z \leq -7.55) \approx 1.00 \end{aligned}$$

4.

$$H_0: \ \mu \geq 8.3 \quad H_1: \ \mu < 8.3$$

Test statistic is

$$\frac{7.6 - 8.3}{1.2/\sqrt{18}} = -2.47$$

P-value comes from t-distribution with d.f.=17. From the table, $.01 < \mathrm{p-value} < .025$, so do not reject H_0, data is not sufficiently strong to reject at the 1% level of significance.

5. There are two tests that must be performed here, one for VSAT and one for QSAT. Both are two-sided tests.
VSAT: $H_0: \ \mu_v = 440, \quad H_1: \mu_v \neq 440$. Test statistic is

$$\frac{|452 - 440|}{110/\sqrt{40}} = 0.69.$$

P-value comes from t-distribution with 39 d.f. Because of the limited t-table in the textbook, we must use the bottom row. $\mathrm{p-value} = 2P(t > 0.69) > .2$ so do not reject H_0. This data is not sufficiently strong to show that the mean VSAT differs from the overall mean. We may be making a Type 2 error. 95% confidence interval is

$$452 \pm (1.95)(110)/\sqrt{40}.$$

QSAT: $H_0: \ \mu_q = 425, \quad H_1: \mu_q \neq 425$. Test statistic is

$$\frac{|448 - 425|}{80/\sqrt{40}} = 1.82.$$

$\mathrm{p-value} = 2P(t > 1.82)$. From the t-table we have $0.05 < \mathrm{p-value} < 0.10$ so reject H_0. The mean QSAT for students who took the course is higher than the overall mean. We may be making a Type 1 error. 95% confidence interval is

$$448 \pm (1.96)(80)/\sqrt{40}.$$

6. $H_0: \ \mu \geq 90, \quad H_1: \ \mu < 90$. Test statistic is

$$\frac{88 - 90}{5.2/\sqrt{20}} = -1.72.$$

P-value comes from t-distribution with 19 d.f. $0.05 < \mathrm{p-value} < 0.10$ so do not reject H_0 at the 5% level of significance. The actual p-value is 0.05084, so rounding errors may give you a different p-value. This is an example of why you should show your work.
$H_0: \ \sigma \leq 3, \quad H_1: \ \sigma > 3$. Test statistic is

$$\frac{(19)(5.2^2)}{3^2} = 57.08.$$

P-value comes from chi-square distribution with 19 d.f. $\mathrm{p-value} < .005$ so reject H_0. Standard for s.d. is not met.