In an experiment, we observe $Y$, whose conditional density on another random variable $X$ is given by

$$f_{Y|X}(y|X = x) = \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp \left( -\frac{(y-x)^2}{2\sigma_A^2} \right).$$

If $f_X(x) = \frac{1}{2} \delta(x - 1) + \frac{1}{2} \delta(x + 1)$,

1. What is $f_{Y,X}(y, x)$?

$$f_{Y,X}(y, x) = f_{Y|X}(y|X = x) f_X(x)$$

$$= \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp \left( -\frac{(y-x)^2}{2\sigma_A^2} \right) \left\{ \frac{1}{2} \delta(x - 1) + \frac{1}{2} \delta(x + 1) \right\}$$

$$= \frac{1}{2\sqrt{2\pi\sigma_A^2}} \exp \left( -\frac{(y-1)^2}{2\sigma_A^2} \right) \delta(x - 1) + \frac{1}{2\sqrt{2\pi\sigma_A^2}} \exp \left( -\frac{(y+1)^2}{2\sigma_A^2} \right) \delta(x + 1)$$

2. What is $f_Y(y)$?

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,X}(y, x) f_X(x) dx = \frac{1}{2\sqrt{2\pi\sigma_A^2}} \exp \left( -\frac{(y-1)^2}{2\sigma_A^2} \right) + \frac{1}{2\sqrt{2\pi\sigma_A^2}} \exp \left( -\frac{(y+1)^2}{2\sigma_A^2} \right)$$

3. What is $f_{X|Y}(x|Y = y)$?

$$f_{X|Y}(x|Y = y) = \exp \left( -\frac{(y-1)^2}{2\sigma_A^2} \right) \delta(x - 1) + \exp \left( -\frac{(y+1)^2}{2\sigma_A^2} \right) \delta(x + 1)$$

$$= \exp \left( -\frac{(y-1)^2}{2\sigma_A^2} \right) + \exp \left( -\frac{(y+1)^2}{2\sigma_A^2} \right)$$

4. Is $X$ when condition on $Y$ a discrete random variable? (Hint: Use (3))

Yes, since $X$ can only take on 2 values (+1 and -1) due to the presence of delta functions at those points. Note that $P(x = 1|Y = y) = \frac{\exp \left( \frac{(y-1)^2}{2\sigma_A^2} \right)}{\exp \left( \frac{(y-1)^2}{2\sigma_A^2} \right) + \exp \left( \frac{(y+1)^2}{2\sigma_A^2} \right)}$ and $P(x = -1|Y = y) = \frac{\exp \left( \frac{(y+1)^2}{2\sigma_A^2} \right)}{\exp \left( \frac{(y-1)^2}{2\sigma_A^2} \right) + \exp \left( \frac{(y+1)^2}{2\sigma_A^2} \right)}.$